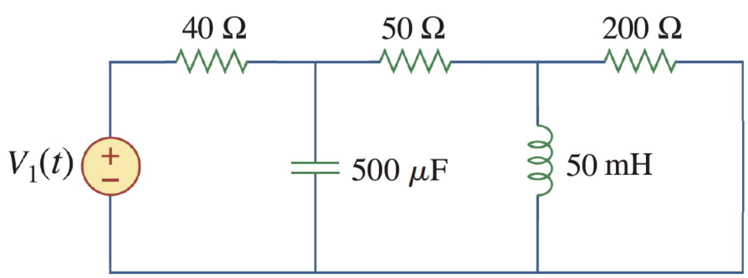
Generic RLC circuit analysis

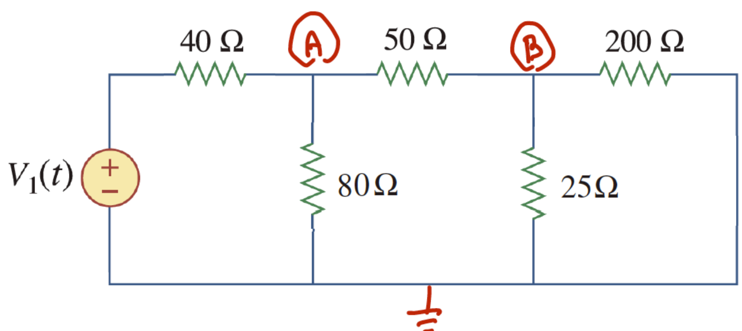
We just saw the two new devices, capacitors and inductors, with their respective voltage-to-current relationships

$$i_C(t) = C\frac{dv_C(t)}{dt} \qquad\qquad i_L(t) = \frac{1}{L}\int^{t} v_L(s)ds$$

A natural question is how such devices impact our ability to analyze circuits. As an example, consider this configuration



How do we start? Well, if instead of the capacitor and inductor we had resistors, as in this figure

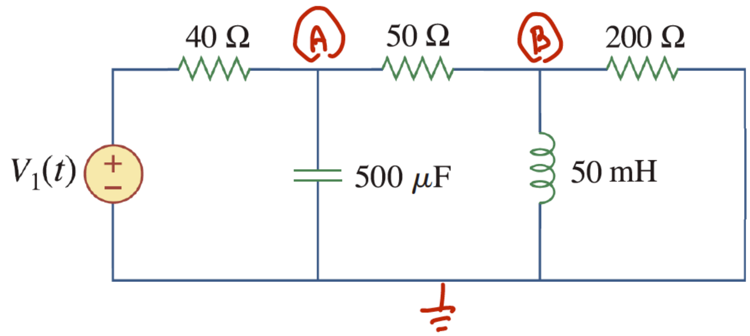


then you would probably use node analysis (as suggested by my marks in red). Specifically, you could write the two node equations

$$\frac{A - V_1(t)}{40} + \frac{A}{80} + \frac{A - B}{50} = 0$$

$$\frac{B}{200} + \frac{B}{25} + \frac{B - A}{50} = 0$$

and then solve using standard algebraic methods. So, let's start the same way on the original RLC (resistor and inductor and capacitor) circuit. Marking the nodes in the same way



We can write the first node equation (for A) as

$$\frac{A(t) - V_1(t)}{40} + i_C(t) + \frac{A(t) - B(t)}{50} = 0$$

in which I have included time notation for all of the terms and just written the capacitor current (defined as downwards) as $i_C(t)$ since I cannot use Ohm's Law on that branch. However, we can use the definition for the capacitor current in terms of its voltage $A(t)$ to get the first node equation as

$$\frac{A(t) - V_1(t)}{40} + 5 \times 10^{-4} \frac{dA(t)}{dt} + \frac{A(t) - B(t)}{50} = 0$$

Similarly, the node equation at B needs the current for the inductor in terms of its voltage $B(t)$

$$\frac{B(t)}{200} + \frac{1}{0.05} \int^{t} B(u) du + \frac{B(t) - A(t)}{50} = 0$$

Since this has an integral in it, we take the time derivative of both sides to yield a second differential equation

$$\frac{1}{200} \frac{dB(t)}{dt} + 20\, B(t) + \frac{1}{50} \frac{dB(t)}{dt} - \frac{1}{50} \frac{dA(t)}{dt} = 0$$

At this point, reorganizing the terms and doing a little algebra to get the coefficients of the derivative terms to equal one (a standard practice), we have the two differential equations

$$\frac{dA(t)}{dt} + 90\, A(t) - 40\, B(t) = 50\, V_1(t)$$

and

$$\frac{dB(t)}{dt} + 800\, B(t) - 0.8 \frac{dA(t)}{dt} = 0$$

We note several things about these equations:

- They are both linear (i.e. the terms with $A(t)$ or $B(t)$ are just linear functions, no squares or anything else) with constant coefficients.
- They are coupled (both equations include terms with both with $A(t)$ and $B(t)$).

As such, lots is known on their joint solution (finding equations for $A(t)$ and $B(t)$ and hopefully you have either seen this already or are seeing it now in MTH 244). Let's quickly summarize the process.

Step 1 – combining into a single differential equation

As a first step, we usually combine the two equations into a single equation in terms of just $A(t)$ or $B(t)$. The process is similar to what we do in simultaneous equations when we eliminate variables by multiplying the equations by the coefficients of the other equations and adding. For this case, let's rewrite the two differential equations using the "D" operator to represent differentiation (dropping the explicit time notation) as

$$(D + 90)\, A - 40\, B = 50\, V_1$$

$$-0.8DA + (\, D + 800)\, B = 0$$

To eliminate B we multiply the first equation by $(\, D + 800)$, multiply the second by 40, and then add the results. This is

$$(\, D + 800)(D + 90)\, A - 40(\, D + 800)\, B - 32DA + 40(\, D + 800)\, B = 50(\, D + 800)\, V_1$$

Simplifying

$$(D^2 \ + 858D + 72{,}000)\, A = 50(\, D + 800)\, V_1$$

Returning to differential notation, this is

$$\frac{d^2A(t)}{dt^2} + 858\ \frac{dA(t)}{dt} + 72{,}000\ \ A(t) = 50\ \frac{dV_1(t)}{dt} + 40{,}000\ \ V_1(t)$$

a second order, linear, constant coefficient differential equation. What about $B(t)$? In a similar fashion, eliminating $A(t)$ yields a second order differential equation for $B(t)$?

$$\frac{d^2B(t)}{dt^2} + 858\ \frac{dB(t)}{dt} + 72{,}000\ \ B(t) = 40\ \frac{dV_1(t)}{dt}$$

We note (!) that the coefficients of $B(t)$ and its derivatives on the left-hand side of this second result are exactly the <u>same</u> as those in the equation for $A(t)$. Further, this same thing happens no matter what voltage or current we are solving for in our circuit; the left-hand side of the differential equation is always the same for all circuit voltages and currents and depends only upon the circuit configuration and the component values (not any applied sources). We will consider this fact again later in the semester.

Step 2 – solving this single variable differential equation

As discussed in math classes on differential equations, the solution to a linear, constant coefficient differential equation consists of two parts, the homogeneous and particular solutions. For this discussion, let's focus on finding $B(t)$

$$\frac{d^2B(t)}{dt^2} + 858 \ \frac{dB(t)}{dt} + 72{,}000 \ B(t) = Y(t)$$

in which $Y(t) = 40 \ \frac{dV_1(t)}{dt}$ is the right-hand side and is not a function of $B(t)$. The general solution to such a differential equation is of the form

$$B(t) = B_{homogeneous}(t) + B_{particular}(t)$$

- The homogeneous part of the result is the solution to the equation without the right-hand side

  $$\frac{d^2B(t)}{dt^2} + 858 \ \frac{dB(t)}{dt} + 72{,}000 \ B(t) = 0$$

  It is well known that this equation is only satisfied by exponential functions of the form

  $$B_{homogeneous}(t) = ae^{st}$$

  in which a is an arbitrary constant and s must satisfy the quadratic equation formed from the coefficients of the differential equation

  $$s^2 + 858\, s + 72{,}000 \ = 0$$

  Since this quadratic has exactly two roots (-94.3 and -764, to 3 significant digits), then the general form of the homogeneous solution must take both into account, each with its own arbitrary coefficient. In general, since we started with a differential equation with real coefficients, we know that the roots are either both real or they form a complex conjugate pair. In this case, with distinct real roots, the solution is

  $$B_{homogeneous}(t) = a_1e^{-94.3t} + a_2e^{-764t}$$

  In our circuit problems we have the added feature that the coefficients in the differential equations are always non-negative, so the real part of both roots is always negative and the homogeneous solution always trends toward zero exponentially quickly as time increases. This leads to our calling this portion of the solution the "*transient*" part in that it disappears as time increases. And for many circuits, this portion of the solution disappears quite quickly (e.g. in much, much less than a second as in this case).

Finally, the full solution for this part of the problem requires finding the multiplicative constants, a_1 and a_2. The usual approach requires forcing known conditions on the circuit at some point in time (for example, using knowledge of the capacitor voltage and the inductor current at time equal to zero). We will return to this issue later in the semester and focus our attention for this moment on the particular solution.

- The particular part of the result for $B(t)$ depends upon the functional form of the right-hand side of the differential equation, written above as $Y(t)$. In differential equation courses this is usually accomplished by resorting to a table such as this one that I found online:

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$ $= \boldsymbol{K\cos(\omega t + \theta)}$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Specifically, you find the functional form of your right-hand side in the first column and use the form in the second column with unknown parameters (the K's). The unknowns are found by plugging the particular solution back into the original differential equation (taking derivatives as necessary) and equating the result to the known right-hand side, $Y(t)$.

For ELE 212 we are primarily interested in only two of these cases:

- A DC source: $V_1(t) = V_s$ so that $Y(t)$ is a constant. In this case the particular solution is also a constant

$$B_{particular}(t) = a$$

  We will return to this case later in the semester.

- A sinusoidal source: $V_1(t) = V_s \cos(\omega t + \phi)$, a sinusoid in which V_s is the amplitude, ω is the frequency measured in radians per second, and ϕ is some phase angle. For our example, let $V_1(t) = 10\cos(500t)$ so that

$$Y(t) = 40\,\frac{dV_1(t)}{dt} = -200{,}000\sin(500t)$$

While the table suggests using a sum of a sine and cosine at frequency $\omega = 500$, it will be more convenient to use the equivalent form

$$B_{particular}(t) = B\cos(500t + \theta)$$

with two parameters, the amplitude B and a phase shift θ. Plugging this into our differential equation and using lots of trigonometry it is possible to solve for both B and θ; the details appear below.

Finally, for both of these cases the particular solution remains non-zero for all time, either a constant or a sinusoid, so we will give this portion another name, calling it the "*steady-state*" part of the solution.

As I hope that you can see, the analysis of a general RLC circuit is quite complicated (both getting the differential equation and then solving it). In class we will next be introducing a much simpler approach for finding the steady-state solution, called the "*phasor method*."

Step 3 – solving for the particular solution for a sinusoidal source

As an example of finding B and θ, let's plug $B(t) = B\cos(500t + \theta)$ into the differential equation for $B(t)$. Using the facts that the first and second derivatives of a cosine are the negative sine and negative cosine, respectively, we have

$$-500^2\, B\cos(500t + \theta) - 858 \times 500\, B\sin(500t + \theta) + 72{,}000\, B\cos(500t + \theta)$$

$$= -200{,}000\sin(500t)$$

To simplify this expression, use the trigonometric identities

$$\sin(500t + \theta) = \sin 500t\cos\theta + \cos 500t\sin\theta$$

and

$$\cos(500t + \theta) = \cos 500t\cos\theta - \sin 500t\sin\theta$$

to yield

$$-250{,}000\, B\cos 500t\cos\theta + 250{,}000\, B\sin 500t\sin\theta - 429{,}000\, B\sin 500t\cos\theta$$

$$-429{,}000\, B\cos 500t\sin\theta + 72{,}000\, B\cos 500t\cos\theta - 72{,}000\, B\sin 500t\sin\theta$$

$$= -200{,}000\sin(500t)$$

Let's gather like terms with the cosine and sine of $500t$

$$[-250{,}000\, B\cos\theta - 429{,}000\, B\sin\theta\; + 72{,}000\, B\cos\theta]\cos 500t$$

$$+[\ 250{,}000\ B\sin\theta - 429{,}000\ B\cos\theta - 72{,}000\ B\sin\theta + 200{,}000\ ]\sin\omega t = 0$$

I realize that this still looks pretty formidable but bear with me. Notice that I've added brackets to isolate terms that are not time varying from those that are.

This equality is required to be satisfied for all values of time; let's consider a few specific ones. Let $t = 0$; since $\cos 0 = 1$ and $\sin 0 = 0$ the requirement is then

$$-178{,}000\ B\cos\theta - 429{,}000\ B\sin\theta = 0$$

Manipulating this

$$-178{,}000\ B\cos\theta = 429{,}000\ B\sin\theta$$

or

$$-\frac{178}{429} = \frac{\sin\theta}{\cos\theta}$$

or

$$\tan\theta = -0.415 \;\;\Rightarrow\;\; \theta = -22.5°$$

and we have the angle! (Strictly, since we will want angles on a full $360°$ range, this would also be satisfied adding $180°$ or $\theta = 157.5°$; more on this below.) Next, consider $t = \frac{\pi}{2}$; since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$ the requirement is

$$178{,}000\ B\sin\theta - 429{,}000\ B\cos\theta + 200{,}000 = 0$$

Since we know the value for θ we can manipulate this expression to

$$B = \frac{200{,}000}{429{,}000\ \cos\theta - 178{,}000\ \sin\theta} = 0.431$$

So the answer is

$$B_{steady-state}(t) = 431\cos(500t - 22.5°)\ mV$$

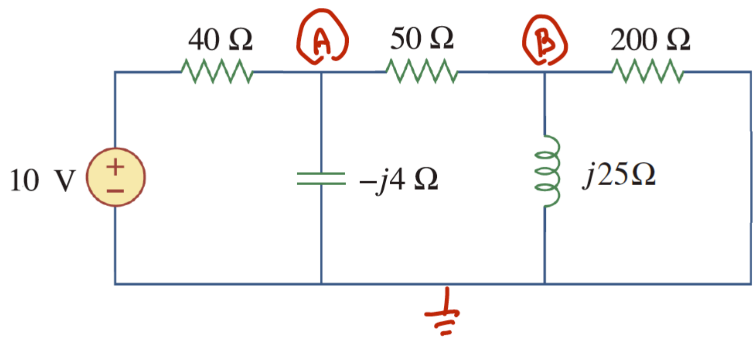
Finally, you might ask, "What about using $\theta = 157.5°$?" In that case we get $B = -0.431$ and since we normally thin of amplitudes as positive, we will choose the original angle.

The Phasor Solution

Let's repeat this problem with phasors.

Step 1 – convert the source to a phasor and the components to impedances using $\omega = 500$ rad/sec

$$V_1 = 10 \qquad Z_L = j\omega L = j25 \qquad Z_C = \frac{1}{j\omega C} = -j4$$



Step 2 – write the node equations using Ohm's Law on impedances

$$\frac{A-10}{40} + \frac{A}{-j4} + \frac{A-B}{50} = 0$$

$$\frac{B}{j25} + \frac{B}{200} + \frac{B-A}{50} = 0$$

Step 3 – solve for B – I'll use Cramer's rule:

$$B = \frac{\begin{vmatrix} \frac{1}{40}+\frac{1}{50}+\frac{1}{-j4} & \frac{1}{4} \\ -\frac{1}{50} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{1}{40}+\frac{1}{50}+\frac{1}{-j4} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{1}{200}+\frac{1}{50}+\frac{1}{j25} \end{vmatrix}} = \frac{\frac{1}{4}\frac{1}{50}}{\left(\frac{1}{40}+\frac{1}{50}+\frac{1}{-j4}\right)\left(\frac{1}{200}+\frac{1}{50}+\frac{1}{j25}\right)-\left(\frac{1}{50}\right)^2}$$

Evaluating this expression

$$B = \frac{0.005}{0.123 + j0.0132} = 0.398 - j0.165 = 0.431\angle -22.5°$$

so

$$B_{steady-state}(t) = 431\cos(500t - 22.5°)\ mV$$

MUCH EASIER!