Appendix A

Simultaneous Equations and Matrix Inversion

In circuit analysis, we often encounter a set of simultaneous equations having the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{A.1}$$

where there are n unknown $x_1, x_2, \ldots, x_n$ to be determined. Equation (A.1) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_2 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{A.2}$$

This matrix equation can be put in a compact form as

$$\mathbf{AX} = \mathbf{B} \tag{A.3}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{A.4}$$

A is a square ($n \times n$) matrix while **X** and **B** are column matrices.

There are several methods for solving Eq. (A.1) or (A.3). These include substitution, Gaussian elimination, Cramer's rule, matrix inversion, and numerical analysis.

A.1 Cramer's Rule

In many cases, Cramer's rule can be used to solve the simultaneous equations we encounter in circuit analysis. Cramer's rule states that the solution to Eq. (A.1) or (A.3) is

$$\boxed{\begin{aligned} x_1 &= \frac{\Delta_1}{\Delta} \\ x_2 &= \frac{\Delta_2}{\Delta} \\ &\vdots \\ x_n &= \frac{\Delta_n}{\Delta} \end{aligned}} \tag{A.5}$$

where the Δ's are the determinants given by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix} \tag{A.6}$$

Notice that Δ is the determinant of matrix **A** and Δ_k is the determinant of the matrix formed by replacing the kth column of **A** by **B**. It is evident from Eq. (A.5) that Cramer's rule applies only when $\Delta \neq 0$. When $\Delta = 0$, the set of equations has no unique solution, because the equations are linearly dependent.

The value of the determinant Δ, for example, can be obtained by expanding along the first row:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{A.7}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} + \cdots + (-1)^{1+n}a_{1n}M_{1n}$$

where the minor M_{ij} is an $(n-1) \times (n-1)$ determinant of the matrix formed by striking out the ith row and jth column. The value of Δ may also be obtained by expanding along the first column:

$$\Delta = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} + \cdots + (-1)^{n+1}a_{n1}M_{n1} \tag{A.8}$$

We now specifically develop the formulas for calculating the determinants of 2 × 2 and 3 × 3 matrices, because of their frequent occurrence in this text. For a 2 × 2 matrix,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{A.9}$$

For a 3 × 3 matrix,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^3 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31}(-1)^4 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \tag{A.10}$$

An alternative method of obtaining the determinant of a 3×3 matrix is by repeating the first two rows and multiplying the terms diagonally as follows.

$$\Delta = \begin{array}{c|ccc|c} & a_{11} & a_{12} & a_{13} & \\ & a_{21} & a_{22} & a_{23} & \\ & a_{31} & a_{32} & a_{33} & \\ - & a_{11} & a_{12} & a_{13} & + \\ - & a_{21} & a_{22} & a_{23} & + \\ - & & & & + \end{array}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21} \quad \textbf{(A.11)}$$

In summary:

The solution of linear simultaneous equations by Cramer's rule boils down to finding

$$x_k = \frac{\Delta_k}{\Delta}, \qquad k = 1, 2, \ldots, n \quad \textbf{(A.12)}$$

where Δ is the determinant of matrix **A** and Δ_k is the determinant of the matrix formed by replacing the kth column of **A** by **B**.

One may use other methods, such as matrix inversion and elimination. Only Cramer's method is covered here, because of its simplicity and also because of the availability of powerful calculators.

You may not find much need to use Cramer's method described in this appendix, in view of the availability of calculators, computers, and software packages such as *MATLAB*, which can be used easily to solve a set of linear equations. But in case you need to solve the equations by hand, the material covered in this appendix becomes useful. At any rate, it is important to know the mathematical basis of those calculators and software packages.

Example A.1

Solve the simultaneous equations

$$4x_1 - 3x_2 = 17, \qquad -3x_1 + 5x_2 = -21$$

Solution:

The given set of equations is cast in matrix form as

$$\begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ -21 \end{bmatrix}$$

The determinants are evaluated as

$$\Delta = \begin{vmatrix} 4 & -3 \\ -3 & 5 \end{vmatrix} = 4 \times 5 - (-3)(-3) = 11$$

$$\Delta_1 = \begin{vmatrix} 17 & -3 \\ -21 & 5 \end{vmatrix} = 17 \times 5 - (-3)(-21) = 22$$

$$\Delta_2 = \begin{vmatrix} 4 & 17 \\ -3 & -21 \end{vmatrix} = 4 \times (-21) - 17 \times (-3) = -33$$

Hence,

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{22}{11} = 2, \qquad x_2 = \frac{\Delta_2}{\Delta} = \frac{-33}{11} = -3$$

Practice Problem A.1

Find the solution to the following simultaneous equations:

$$3x_1 - x_2 = 4, \qquad -6x_1 + 18x_2 = 16$$

Answer: $x_1 = 1.833, x_2 = 1.5.$

Example A.2

Determine $x_1, x_2,$ and x_3 for this set of simultaneous equations:

$$\begin{aligned} 25x_1 - 5x_2 - 20x_3 &= 50 \\ -5x_1 + 10x_2 - 4x_3 &= 0 \\ -5x_1 - 4x_2 + 9x_3 &= 0 \end{aligned}$$

Solution:

In matrix form, the given set of equations becomes

$$\begin{bmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

We apply Eq. (A.11) to find the determinants. This requires that we repeat the first two rows of the matrix. Thus,

$$\Delta = \begin{vmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \end{vmatrix} = \begin{vmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \\ 25 & -5 & -20 \\ -5 & 10 & -4 \end{vmatrix} \begin{matrix} \\ \\ \\ - \; + \\ - \; + \\ - \; + \end{matrix}$$

$$\begin{aligned} &= 25(10)9 + (-5)(-4)(-20) + (-5)(-5)(-4) \\ &\quad - (-20)(10)(-5) - (-4)(-4)25 - 9(-5)(-5) \\ &= 2250 - 400 - 100 - 1000 - 400 - 225 = 125 \end{aligned}$$

Similarly,

$$\Delta_1 = \begin{vmatrix} 50 & -5 & -20 \\ 0 & 10 & -4 \\ 0 & -4 & 9 \end{vmatrix} = \begin{vmatrix} 50 & -5 & -20 \\ 0 & 10 & -4 \\ 0 & -4 & 9 \\ 50 & -5 & -20 \\ 0 & 10 & -4 \end{vmatrix} \begin{matrix} \\ \\ \\ - \; + \\ - \; + \\ - \; + \end{matrix}$$

$$= 4500 + 0 + 0 - 0 - 800 - 0 = 3700$$

$$\Delta_2 = \begin{vmatrix} 25 & 50 & -20 \\ -5 & 0 & -4 \\ -5 & 0 & 9 \end{vmatrix} = \begin{array}{c} \\ \\ \\ - \\ - \\ - \end{array} \begin{vmatrix} 25 & 50 & -20 \\ -5 & 0 & -4 \\ -5 & 0 & 9 \\ 25 & 50 & -20 \\ -5 & 0 & -4 \end{vmatrix} \begin{array}{c} \\ \\ \\ + \\ + \\ + \end{array}$$

$$= 0 + 0 + 1000 - 0 - 0 + 2250 = 3250$$

$$\Delta_3 = \begin{vmatrix} 25 & -5 & 50 \\ -5 & 10 & 0 \\ -5 & -4 & 0 \end{vmatrix} = \begin{array}{c} \\ \\ \\ - \\ - \\ - \end{array} \begin{vmatrix} 25 & -5 & 50 \\ -5 & 10 & 0 \\ -5 & -4 & 0 \\ 25 & -5 & 50 \\ -5 & 10 & 0 \end{vmatrix} \begin{array}{c} \\ \\ \\ + \\ + \\ + \end{array}$$

$$= 0 + 1000 + 0 + 2500 - 0 - 0 = 3500$$

Hence, we now find

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{3700}{125} = 29.6$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{3250}{125} = 26$$

$$x_3 = \frac{\Delta_2}{\Delta} = \frac{3500}{125} = 28$$

Practice Problem A.2

Obtain the solution of this set of simultaneous equations:

$$\begin{aligned} 3x_1 - x_2 - 2x_3 &= 1 \\ -x_1 + 6x_2 - 3x_3 &= 0 \\ -2x_1 - 3x_2 + 6x_3 &= 6 \end{aligned}$$

Answer: $x_1 = 3 = x_3$, $x_2 = 2$.

A.2 Matrix Inversion

The linear system of equations in Eq. (A.3) can be solved by matrix inversion. In the matrix equation $\mathbf{AX} = \mathbf{B}$, we may invert $\mathbf{A}$ to get $\mathbf{X}$, i.e.,

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \tag{A.13}$$

where $\mathbf{A}^{-1}$ is the inverse of **A.** Matrix inversion is needed in other applications apart from using it to solve a set of equations.

By definition, the inverse of matrix **A** satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{A.14}$$

where **I** is an identity matrix. $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{\text{adj }\mathbf{A}}{\det \mathbf{A}} \tag{A.15}$$

where adj **A** is the adjoint of **A** and det **A** = |**A**| is the determinant of **A**. The adjoint of **A** is the transpose of the cofactors of **A**. Suppose we are given an $n \times n$ matrix **A** as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A.16}$$

The cofactors of **A** are defined as

$$\mathbf{C} = \text{cof}\,(\mathbf{A}) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{A.17}$$

where the cofactor c_{ij} is the product of $(-1)^{i+j}$ and the determinant of the $(n-1) \times (n-1)$ submatrix is obtained by deleting the ith row and jth column from **A**. For example, by deleting the first row and the first column of **A** in Eq. (A.16), we obtain the cofactor c_{11} as

$$c_{11} = (-1)^2 \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{A.18}$$

Once the cofactors are found, the adjoint of **A** is obtained as

$$\text{adj}\,(\mathbf{A}) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}^T = \mathbf{C}^T \tag{A.19}$$

where T denotes transpose.

In addition to using the cofactors to find the adjoint of **A**, they are also used in finding the determinant of **A** which is given by

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij}c_{ij} \tag{A.20}$$

where i is any value from 1 to n. By substituting Eqs. (A.19) and (A.20) into Eq. (A.15), we obtain the inverse of **A** as

$$\boxed{\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{|\mathbf{A}|}} \tag{A.21}$$

For a 2×2 matrix, if

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{A.22}$$

its inverse is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{A.23}$$

For a 3 × 3 matrix, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{A.24}$$

we first obtain the cofactors as

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{A.25}$$

where

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$
$$c_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad c_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad c_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$
$$c_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad c_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad c_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{A.26}$$

The determinant of the 3 × 3 matrix can be found using Eq. (A.11). Here, we want to use Eq. (A.20), i.e.,

$$|\mathbf{A}| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \tag{A.27}$$

The idea can be extended $n > 3$, but we deal mainly with 2 × 2 and 3 × 3 matrices in this book.

Example A.3

Use matrix inversion to solve the simultaneous equations

$$2x_1 + 10x_2 = 2, \qquad -x_1 + 3x_2 = 7$$

Solution:

We first express the two equations in matrix form as

$$\begin{bmatrix} 2 & 10 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

or

$$\mathbf{AX} = \mathbf{B} \longrightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 10 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

The determinant of **A** is $|\mathbf{A}| = 2 \times 3 - 10(-1) = 16$, so the inverse of **A** is

$$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 3 & -10 \\ 1 & 2 \end{bmatrix}$$

Hence,

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{16}\begin{bmatrix} 3 & -10 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} -64 \\ 16 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

i.e., $x_1 = -4$ and $x_2 = 1$.

Practice Problem A.3

Solve the following two equations by matrix inversion.

$$2y_1 - y_2 = 4, \quad y_1 + 3y_2 = 9$$

Answer: $y_1 = 3, y_2 = 2$.

Example A.4

Determine x_1, x_2, and x_3 for the following simultaneous equations using matrix inversion.

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ -x_1 + 2x_2 &= 9 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

Solution:
In matrix form, the equations become

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$$

or

$$\mathbf{AX} = \mathbf{B} \longrightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 4 & 1 & -1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$$

We now find the cofactors

$$c_{11} = \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -2, \quad c_{12} = -\begin{vmatrix} -1 & 0 \\ 4 & -1 \end{vmatrix} = -1, \; c_{13} = \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -9$$

$$c_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2, \; c_{22} = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -5, \quad c_{23} = -\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 3$$

$$c_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2, \quad c_{32} = -\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -1, \quad c_{33} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$$

The adjoint of matrix **A** is

$$\text{adj}\,\mathbf{A} = \begin{bmatrix} -2 & -1 & -9 \\ 2 & -5 & 3 \\ -2 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & -2 \\ -1 & -5 & -1 \\ -9 & 3 & 3 \end{bmatrix}$$

We can find the determinant of **A** using any row or column of **A**. Since one element of the second row is 0, we can take advantage of this to find the determinant as

$$|\mathbf{A}| = -1c_{21} + 2c_{22} + (0)c_{23} = -1(2) + 2(-5) = -12$$

Hence, the inverse of **A** is

$$\mathbf{A}^{-1} = \frac{1}{-12}\begin{bmatrix} -2 & 2 & -2 \\ -1 & -5 & -1 \\ -9 & 3 & 3 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{-12}\begin{bmatrix} -2 & 2 & -2 \\ -1 & -5 & -1 \\ -9 & 3 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

i.e., $x_1 = -1, x_2 = 4, x_3 = 2.$

Practice Problem A.4

Solve the following equations using matrix inversion.

$$\begin{aligned} y_1 - y_3 &= 1 \\ 2y_1 + 3y_2 - y_3 &= 1 \\ y_1 - y_2 - y_3 &= 3 \end{aligned}$$

Answer: $y_1 = 6, y_2 = -2, y_3 = 5.$