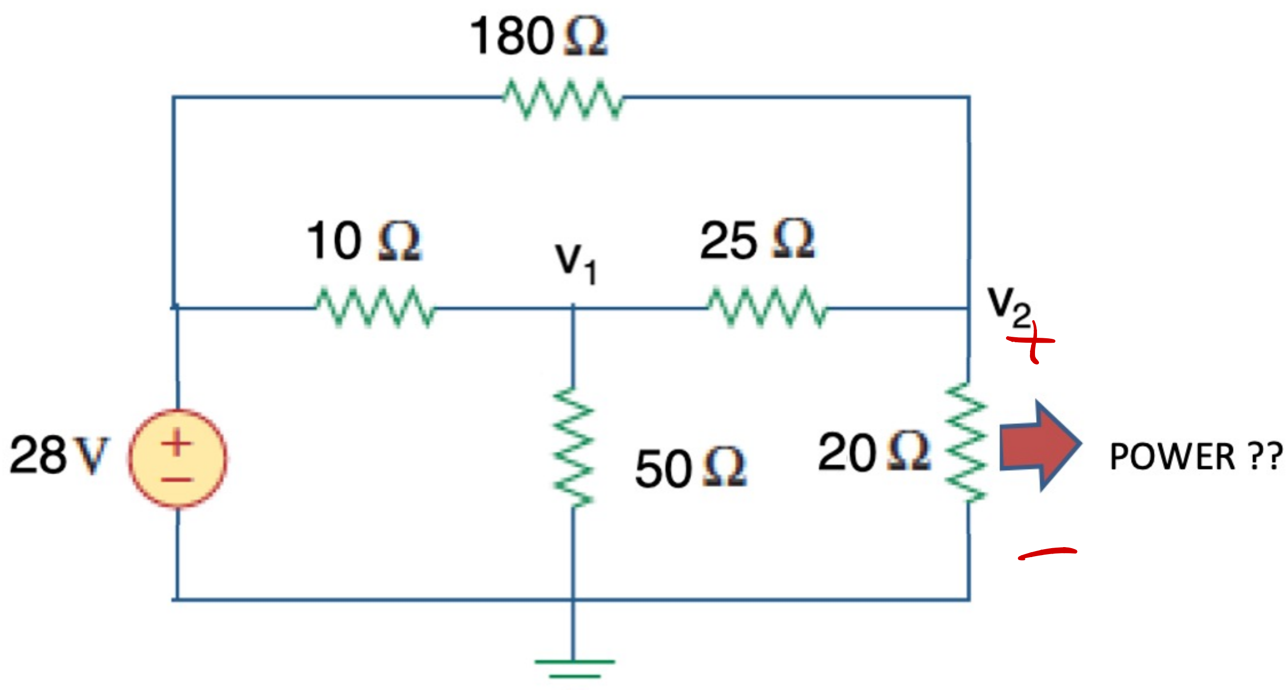
Theorems – 4

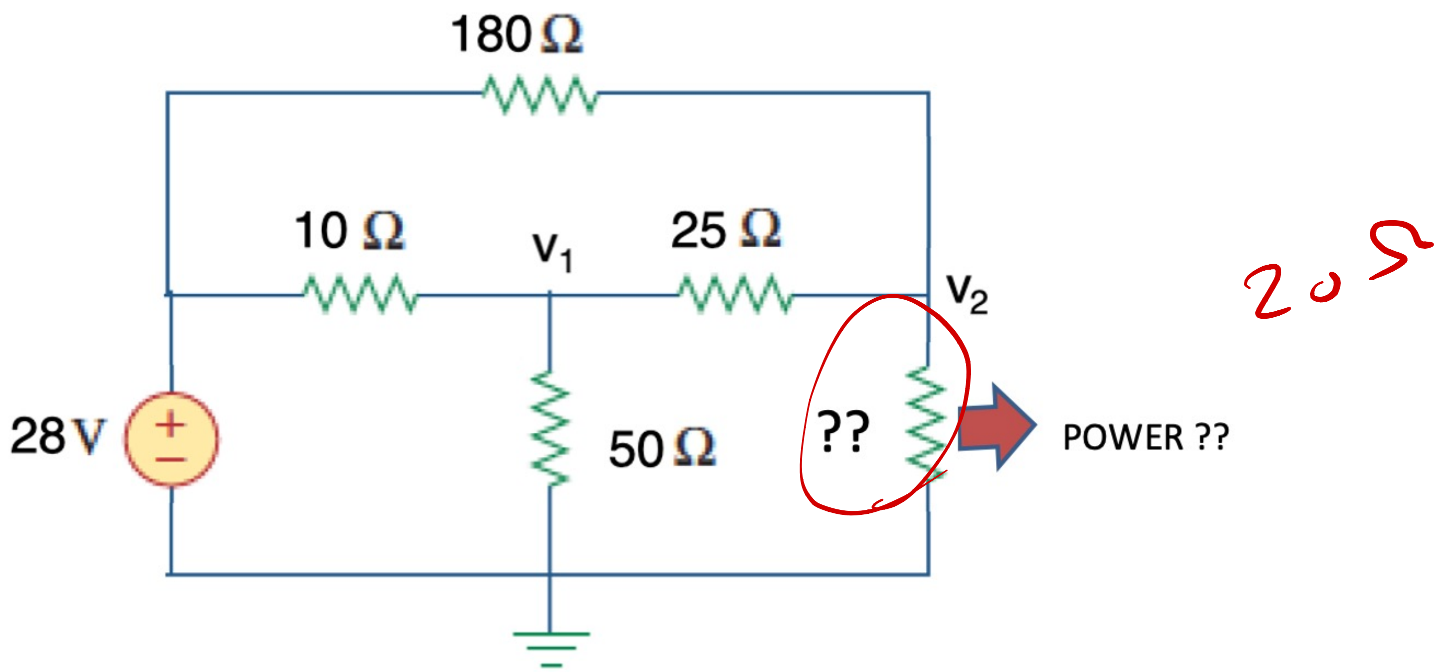
maximum power transfer

Power Transfer

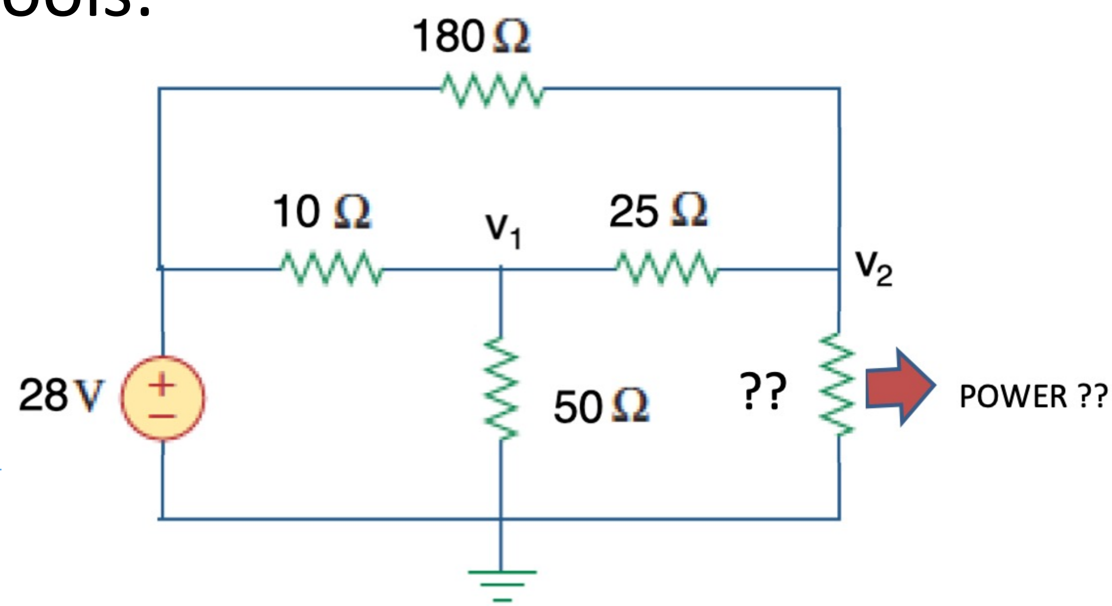


- How much power is dissipated in the 20 Ω resistor?
  - Method: node analysis $\rightarrow v_2 = 10\text{ V}$
  - Power calculation $P = \frac{{v_2}^2}{20} = \frac{10^2}{20} = 5\text{ W}$

- Question – if the resistance was larger/smaller than 20 Ω could it take more power from the circuit?



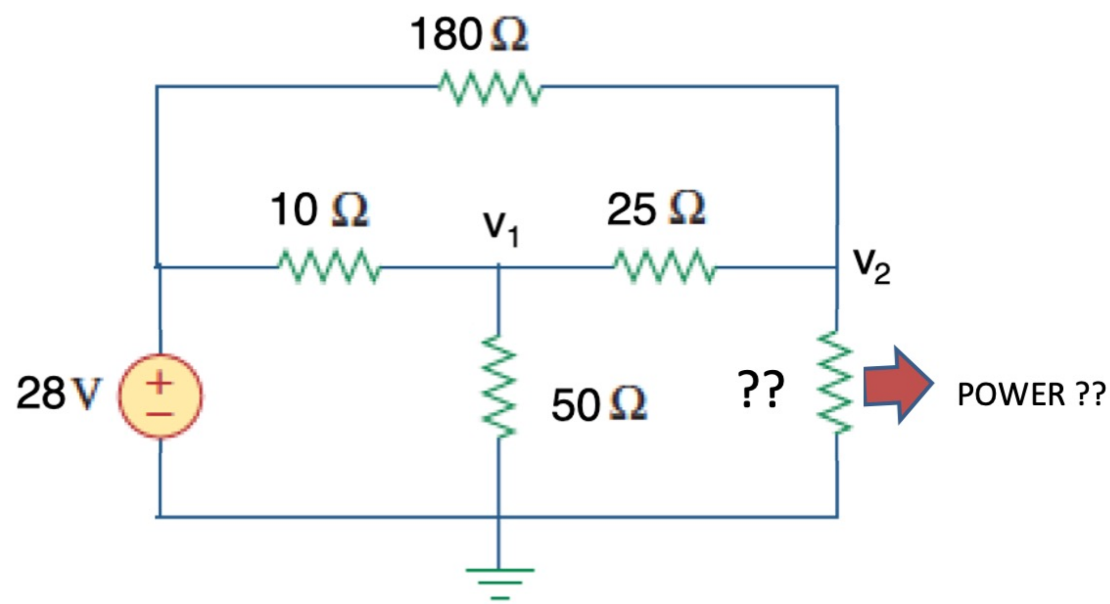
- Approach 1 – solve for power in terms of R
  - MatLab symbolic tools:




```
%% setup problem
syms v1 v2 R
[s1,s2] = solve( v1/50+(v1-28)/10+(v1-v2)/25==0, ...
   v2/R+(v2-v1)/25+(v2-28)/180==0,v1,v2)
pow = s2^2/R;
%% check for R = 20 ohms
subs(s2,R,20)
subs(pow,R,20)
```

```
ans =
10
ans =
5
```

- Optimize



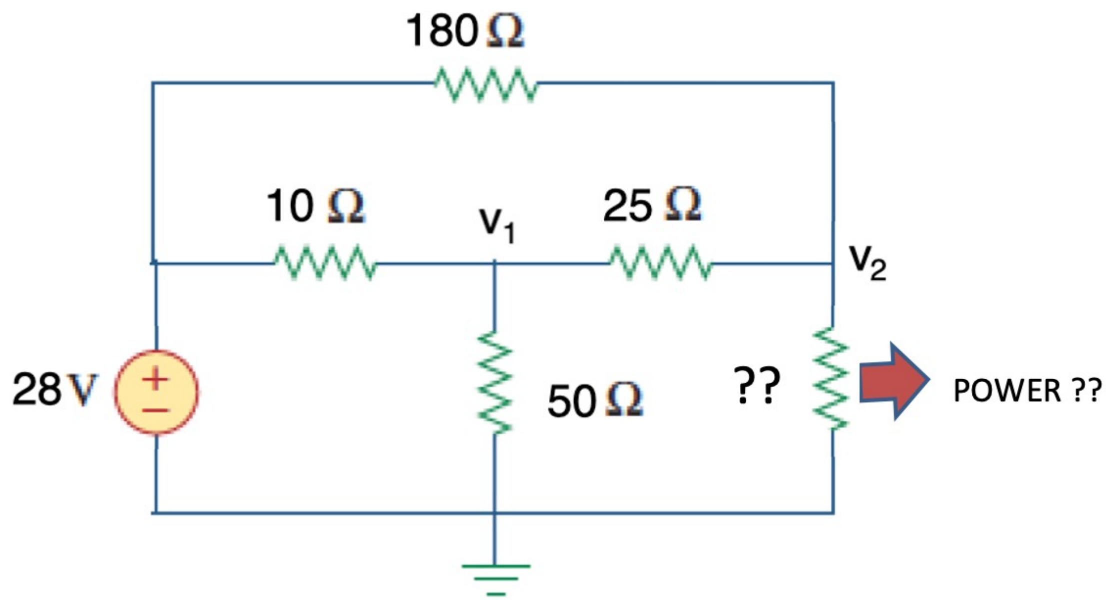

```
% optimize over R
dpow = diff(pow,R);
Rstar = solve(dpow)
eval( subs(pow,R,Rstar) )
```

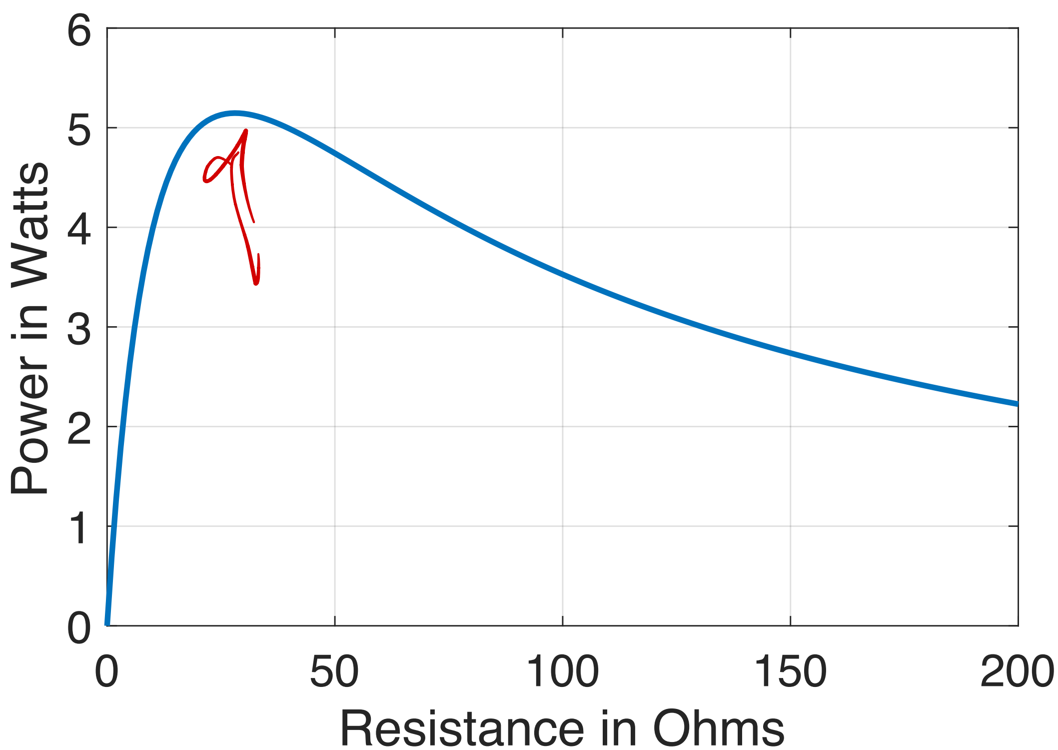
```
Rstar =
225/8
ans =
  5.1467e+00
```

$$R^* = \frac{225}{8} = 28.1\ \Omega \qquad P_{max} = 5.15\ W$$

- What’s going on?

$$P = \frac{148225\,R}{4\,(8R + 225)^2}$$

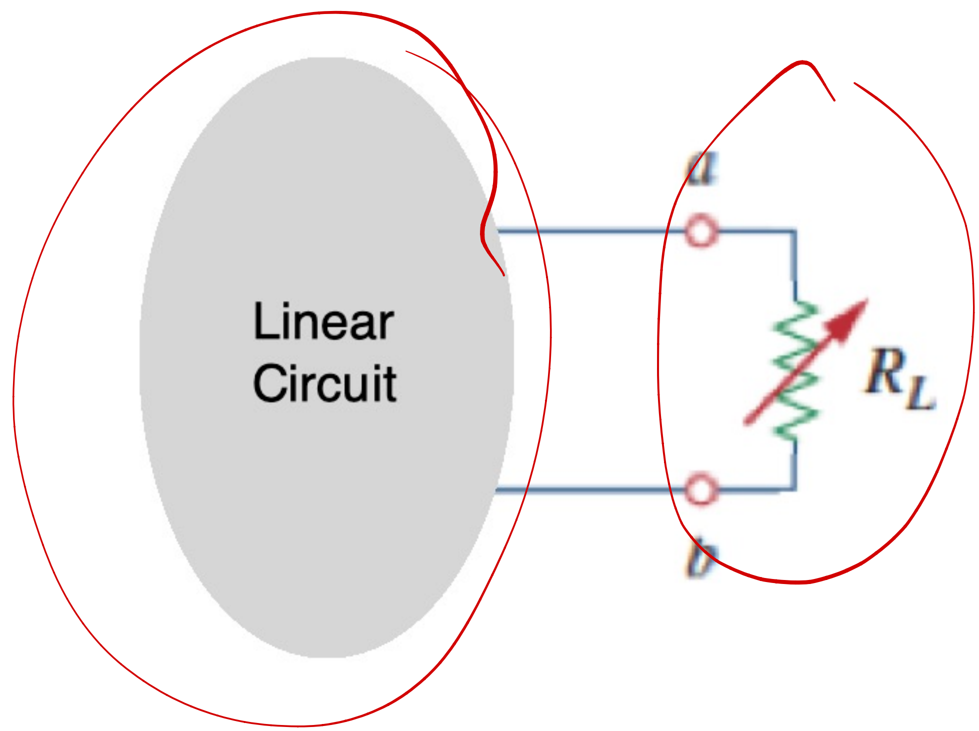




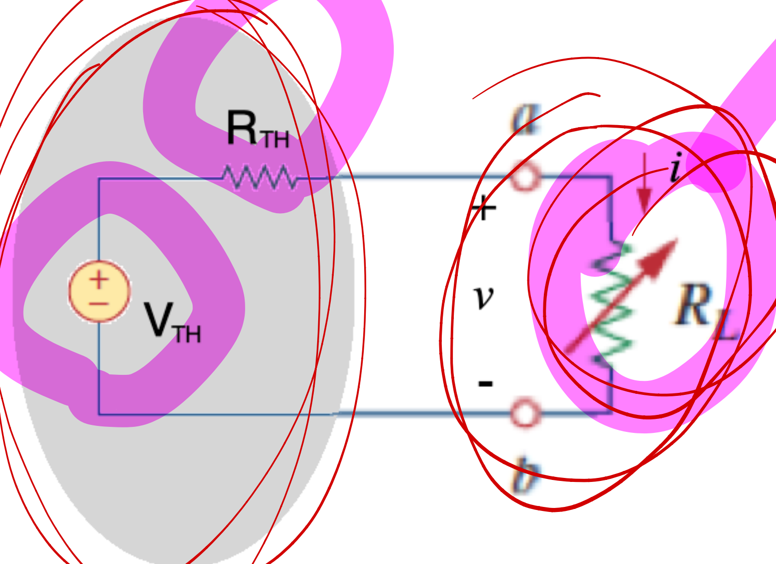

```
Rv = 0:200;
Pv = subs(pow,Rv);
plot(Rv,Pv)
```

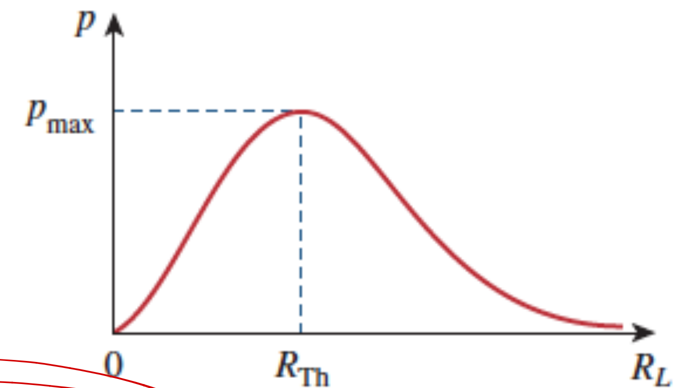
Maximum Power Transfer

- Consider connecting a "load" resistance, R_L, across two points of a circuit
- What happens as it varies?
  - Current
  - Voltage
  - Power



- Use the Thevenin model





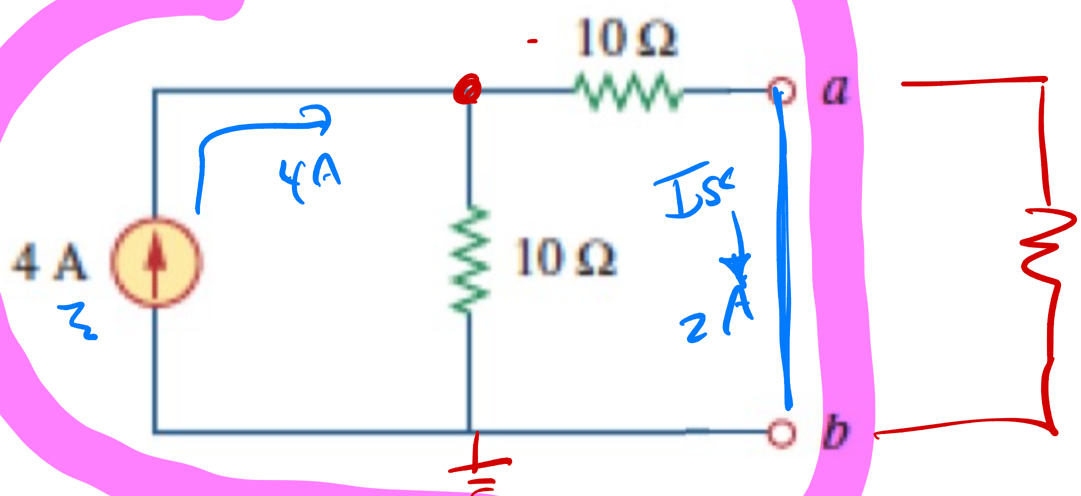
$$v = \frac{R_L}{R_L + R_{th}} v_{th}$$

$$p = \frac{v^2}{R_L} = \frac{R_L}{(R_L + R_{th})^2} v_{th}^2$$

- $\frac{\partial p}{\partial R_L} = 0$ yields a max of $P_{max} = \frac{V_{th}^2}{4R_{th}}$ when $R_L = R_{th}$

Example: find a load resistance to dissipate maximum power

$20\ \Omega, 20\ W$



$R_L = 20\ \Omega$

$P = \frac{40^2}{4 \cdot 20}$

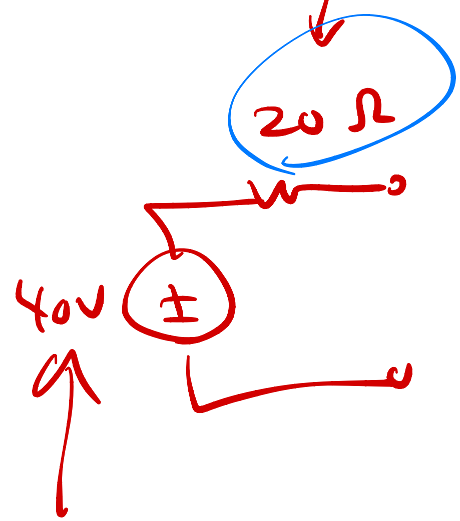
Thevenin

oc: $\frac{A}{10} - 4 = 0 \Rightarrow A = 40$

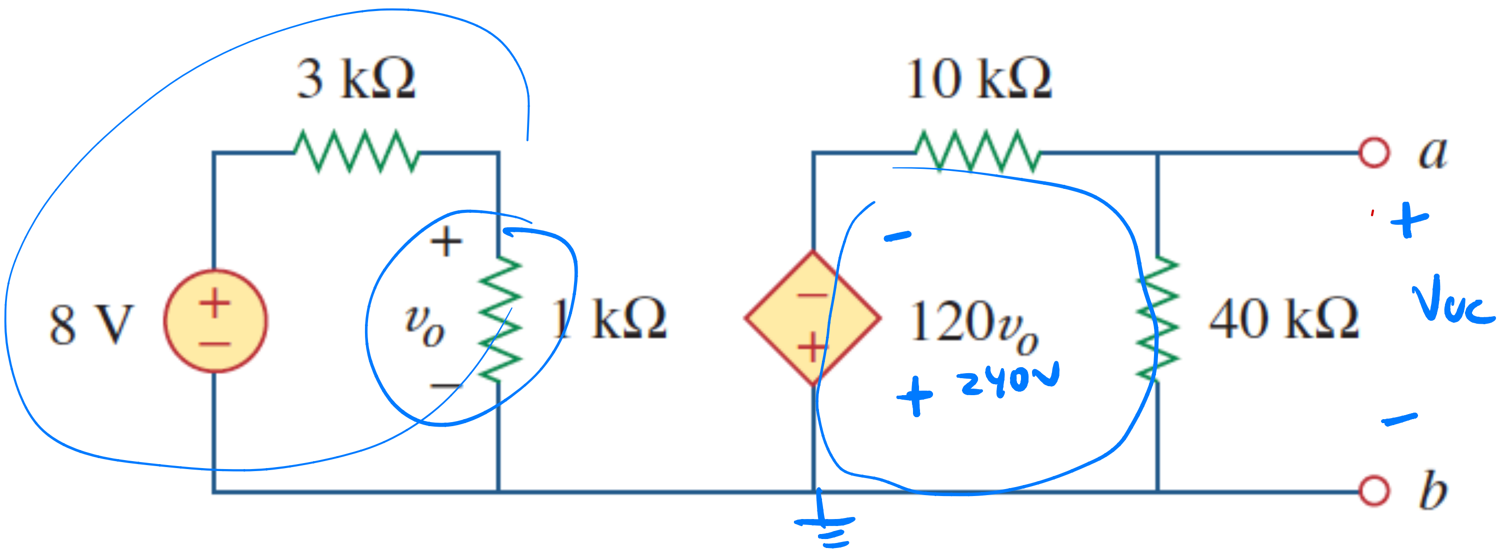
$V_{oc} = 40\ V$

sc: $I_{sc} = 2\ A$

$R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{40}{2} = 20$



Example: find the load that dissipates maximum power

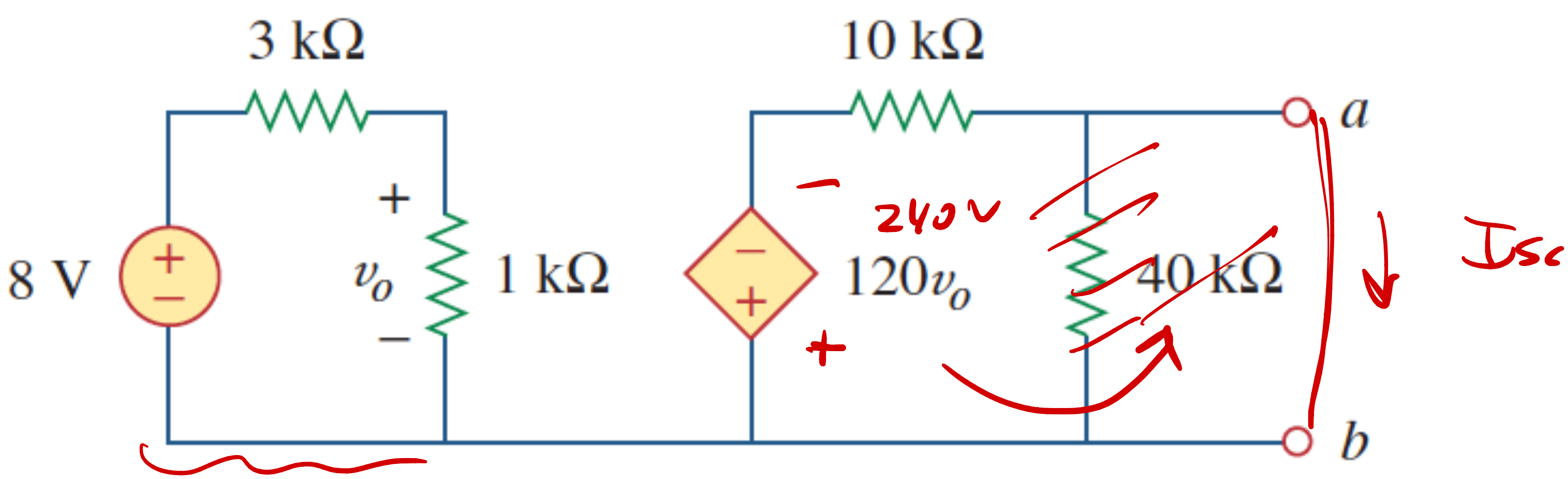


Voc :

$V_o = \frac{1}{4} \cdot 8 = 2V$

$V_{oc} = -240 \cdot \frac{40}{50} = -192\ V$

I_{sc}



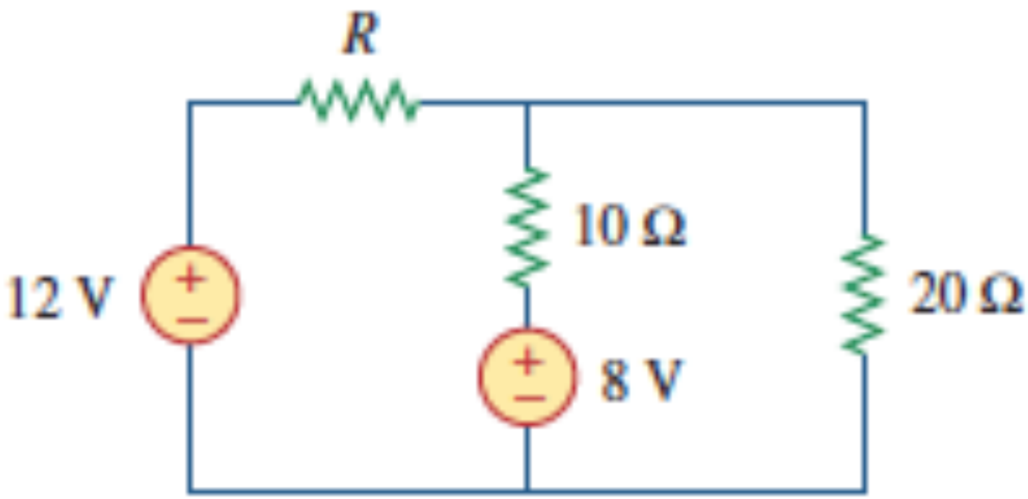
I_{sc} : $v_0 = 2v$ $\qquad i_{sc} = -\frac{240}{10000} = -24\ mA$

$R_{TH} = \frac{-192\,v}{-24\ mA} =$

$P = \frac{(-192)^2}{4 \cdot 8000}\ w$

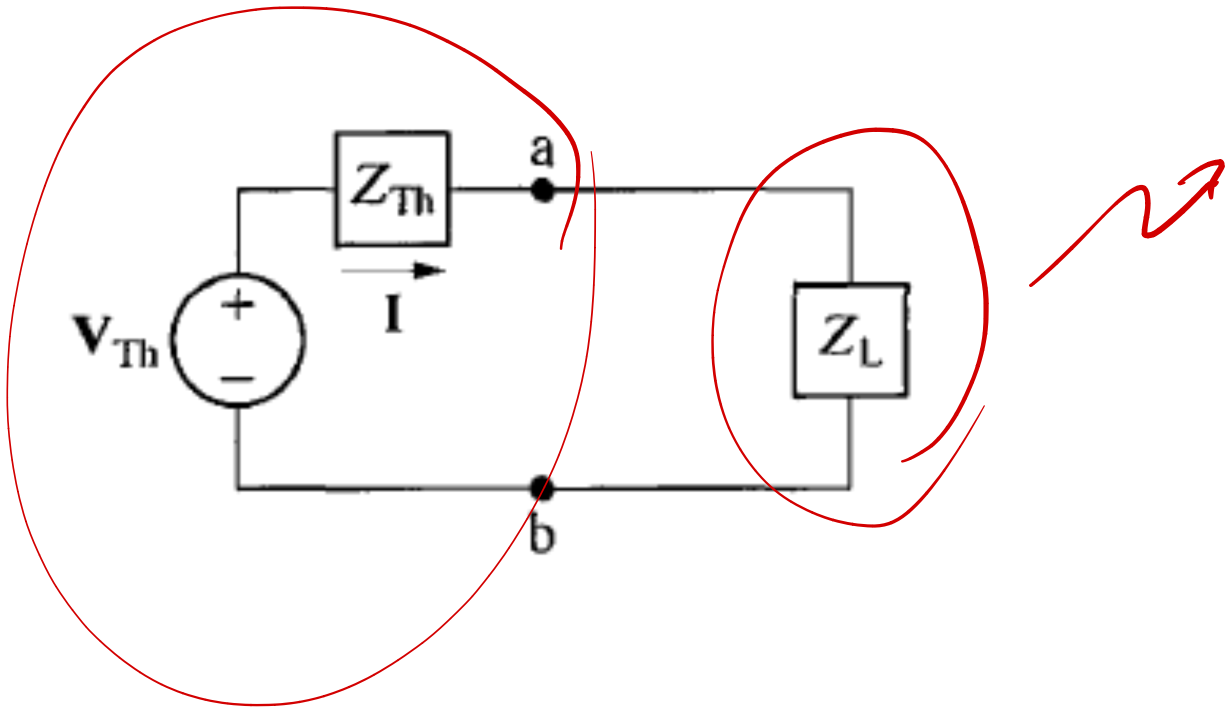
8 kΩ

Example (trick): find R to maximum the power delivered to the 10 ohm resistor



Maximum AC Power

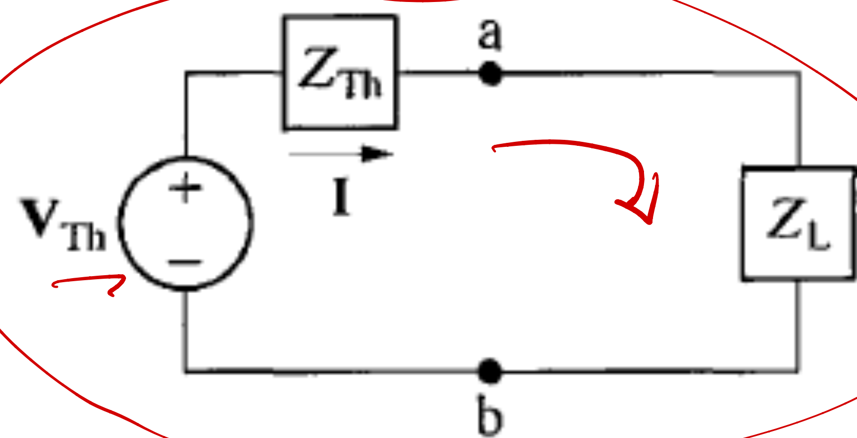
- Given a phasor Thevenin model, how do we get maximum power to Z_L?



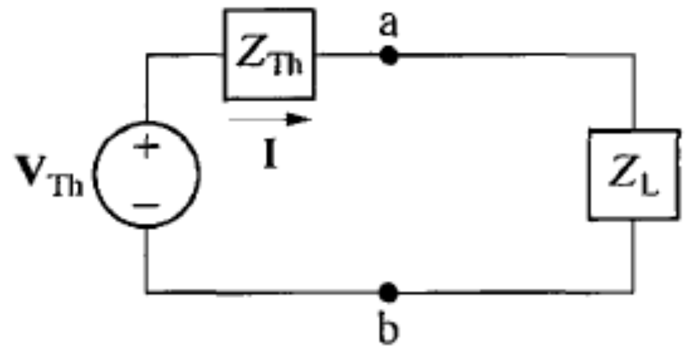
- For sinusoidal sources and RLC circuits, power is

$$S = \frac{|\mathbf{I}|^2}{2} Z_L \qquad P = \frac{|\mathbf{I}|^2}{2} R_L$$

- For our problem



$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_{Th} + Z_L} = \frac{\mathbf{V}_{Th}}{(R_{Th} + R_L) + j(X_{Th} + X_L)}$$



- So

$$P = \frac{|\mathbf{I}|^2}{2} R_L = \frac{1}{2} \frac{|\mathbf{V}_{Th}|^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

- We can optimize this using calculus

- How depends upon which parameters we can change

$$P = \frac{1}{2} \frac{|\mathbf{V}_{\mathrm{Th}}|^2 \mathrm{R_L}}{(\mathrm{R_{Th}} + \mathrm{R_L})^2 + (\mathrm{X_{Th}} + \mathrm{X_L})^2}$$

- Example 1 (unusual): both R_L and X_L are free to choose

$$\frac{\partial P}{\partial X_L} = 0 \qquad \frac{\partial P}{\partial R_L} = 0$$

$$Z_L = Z_{\mathrm{Th}}{}^*$$

$$P_{max} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2}{8R_{Th}}$$

$$P = \frac{1}{2}\frac{|\mathbf{V}_{\mathrm{Th}}|^2 \mathrm{R_L}}{(\mathrm{R_{Th}} + \mathrm{R_L})^2 + (\mathrm{X_{Th}} + \mathrm{X_L})^2}$$

- Example 2 (more common): X_L is fixed, but R_L is free to choose

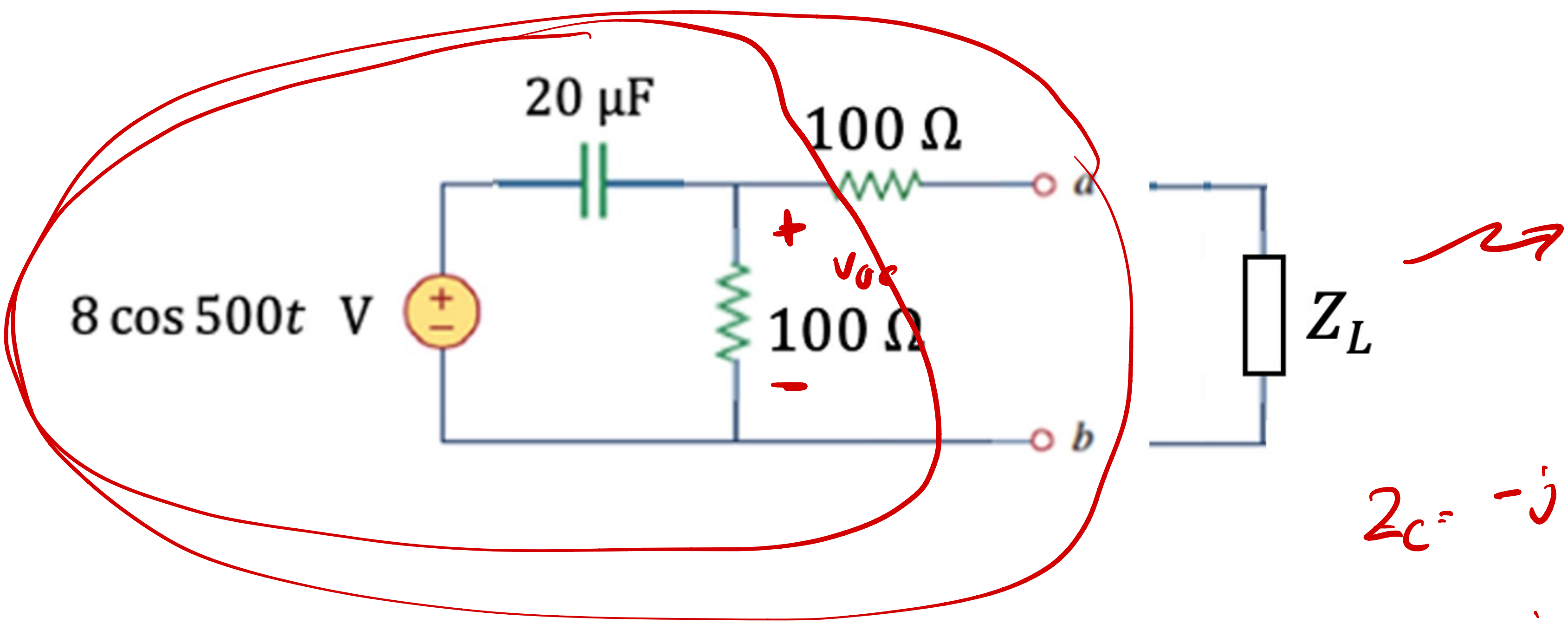
$$\frac{\partial P}{\partial R_L} = 0$$

$$\boxed{\begin{gathered}\mathrm{R_L} = \sqrt{\mathrm{R_{Th}}^2 + (\mathrm{X_{Th}} + \mathrm{X_L})^2} \\ P_{max} = \cdots\end{gathered}}$$

$$P = \frac{1}{2}\frac{|\mathbf{V}_{\mathrm{Th}}|^2 \mathrm{R_L}}{(\mathrm{R_{Th}} + \mathrm{R_L})^2 + (\mathrm{X_{Th}} + \mathrm{X_L})^2}$$

- Other scenarios:
  - Fixed angle on Z_L
  - Limits on $\mathrm{R_L}$ and $\mathrm{X_L}$

Example: find Z_L to maximize the power transfer



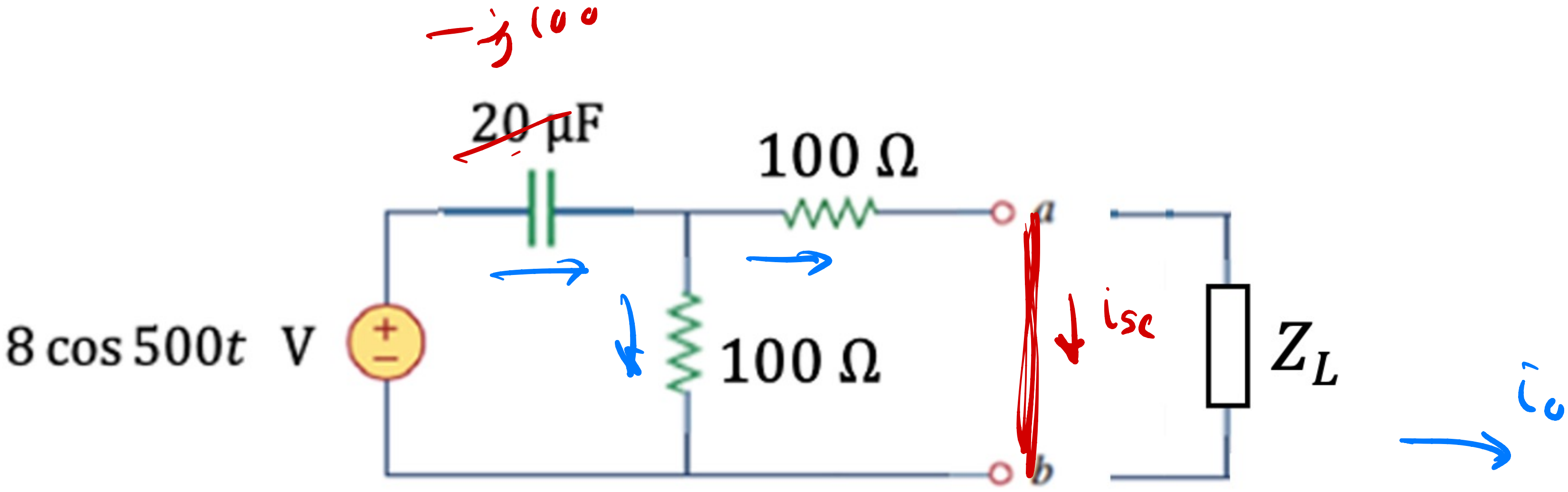
$$Z_C = -j\frac{1}{\omega C} = -j\frac{10^6}{500 \cdot 20} = -j\frac{10^6}{10^4} = -j100$$

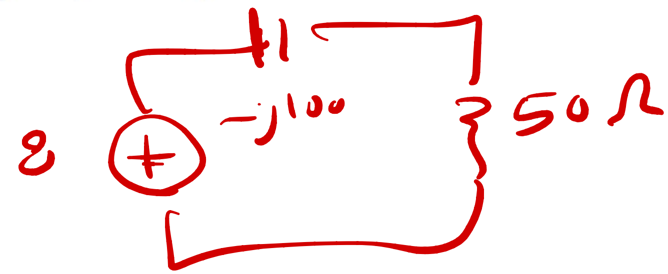
$$V_{oc}: = \frac{100}{100 - j100} \cdot 8$$

$$= \frac{8}{1-j} \cdot \frac{1+j}{1+j} = \boxed{4(1+j)}$$

$$= 4\sqrt{2} \angle 45^\circ$$

$$\boxed{V_{Th} = 4\sqrt{2}\cos(500t + 45^\circ)\,V}$$





I_{sc}:

$$I_o = \frac{8}{50 - j100}$$

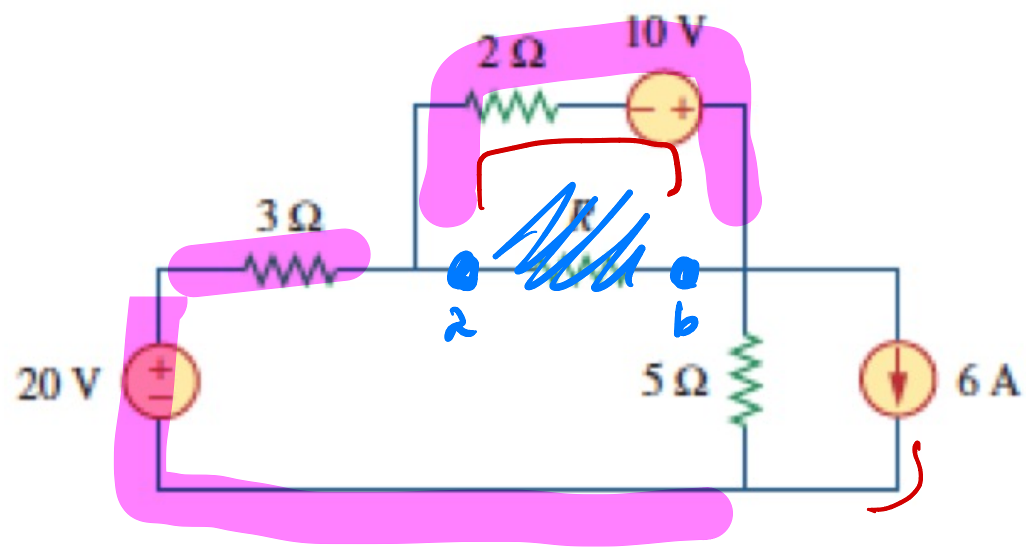
$$I_{sc} = \frac{4}{50 - j100}$$

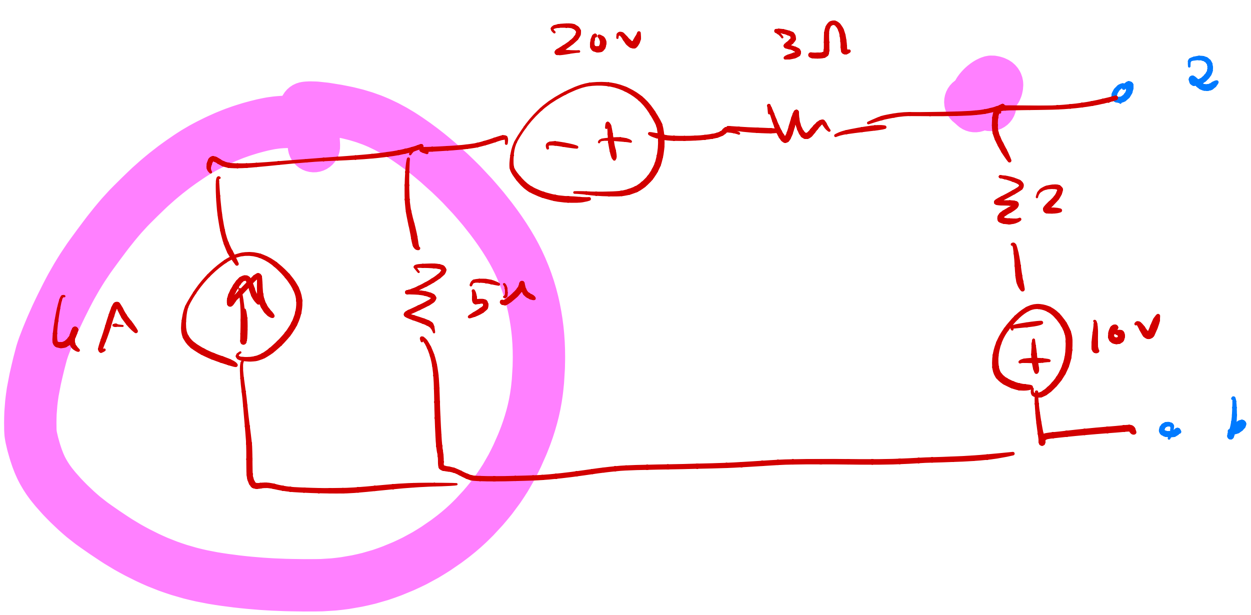
$$Z_{Th} = \frac{\cancel{4}(1+j)}{\frac{\cancel{4}}{50-j100}} = (1+j)(50-j100) = \begin{matrix} 50 + j50 \\ -j100 + 100 \end{matrix} = 150 - j50$$

Since $V_{Th} = 4\sqrt{2}\ \cos(500t + 45°)\ V$ and $Z_{Th} = 150 - j50\ \Omega$,
then $Z_{Th} = 150 + j50\ \Omega = 150\ \Omega, 0.1\ H$, and $P = 26.7\ mW$

$1.6\ \Omega, \frac{5}{8}\ W$

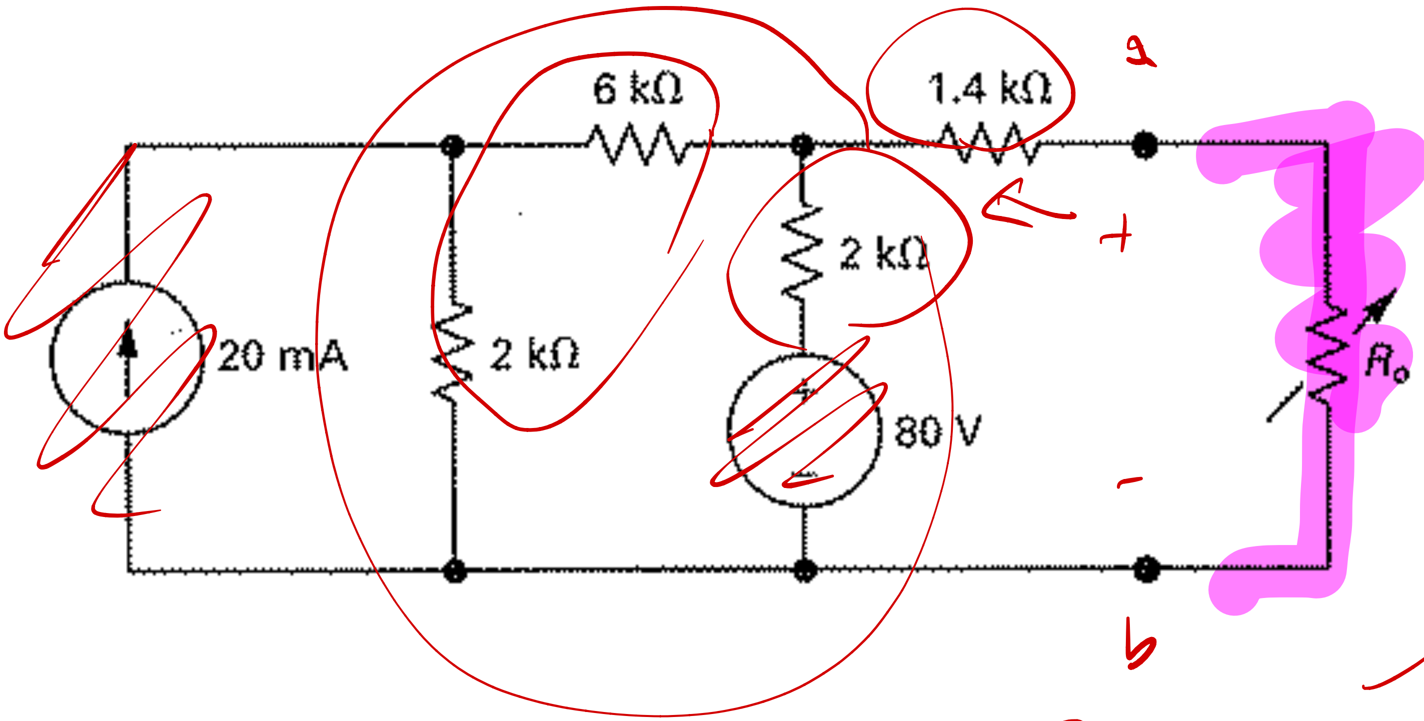
Practice problem: maximize the power to R





$3\ k\Omega, 468\ mW$

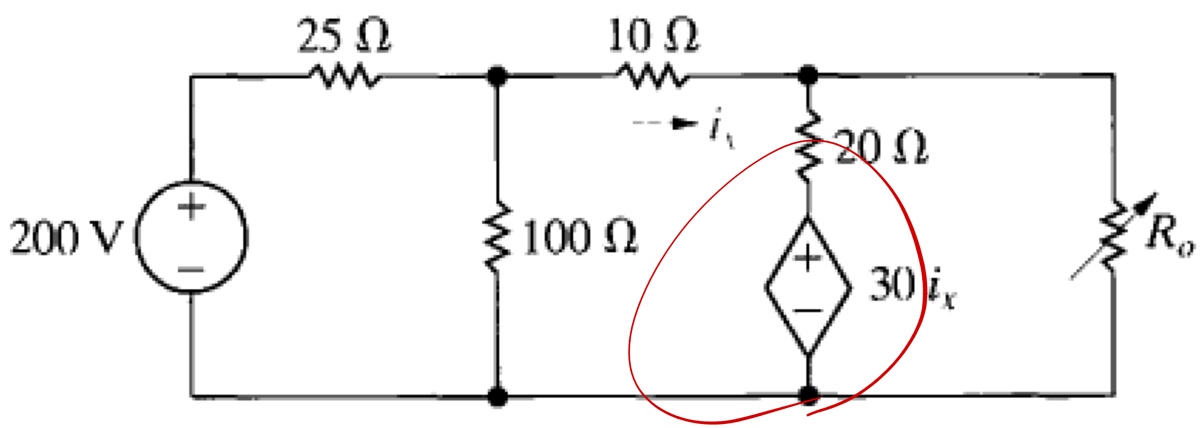
Practice problem: maximize the power to R_o



$1.4 + 1.6 = 3K = R_{th}$

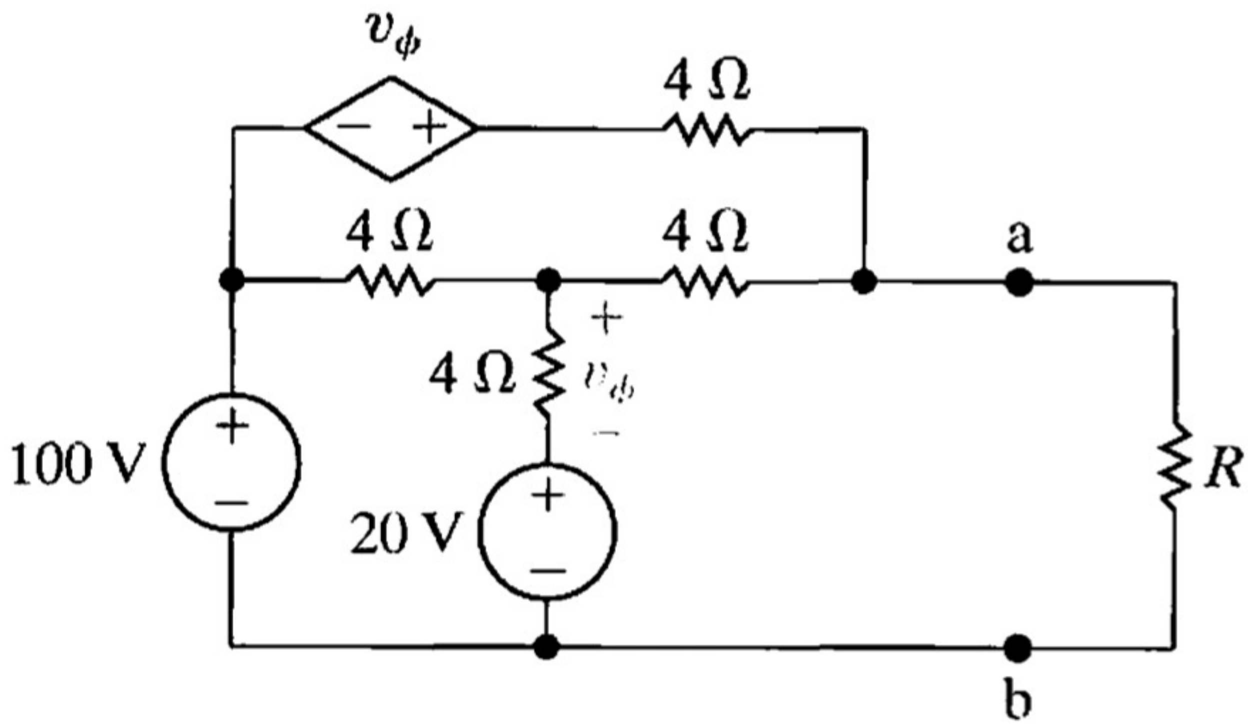
$7.5\ \Omega, 333\ W$

Practice problem: maximize the power to R_o



$3\ \Omega, 1.2\ kW$

Practice problem: maximize the power to R



$4\ k\Omega, 9\ mW$

Practice problem: maximize the power to R_L