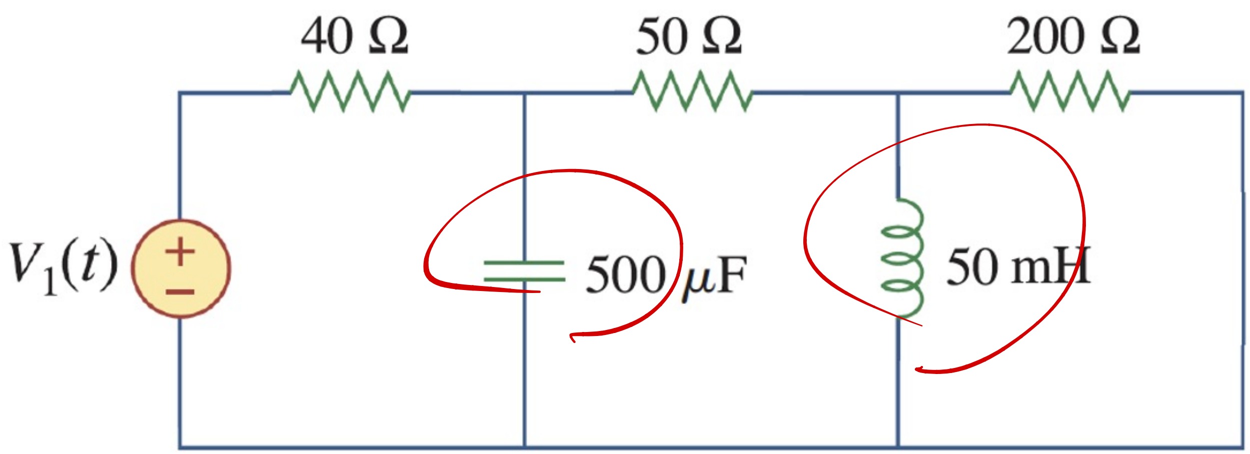
Phasors – 2

RLC circuits

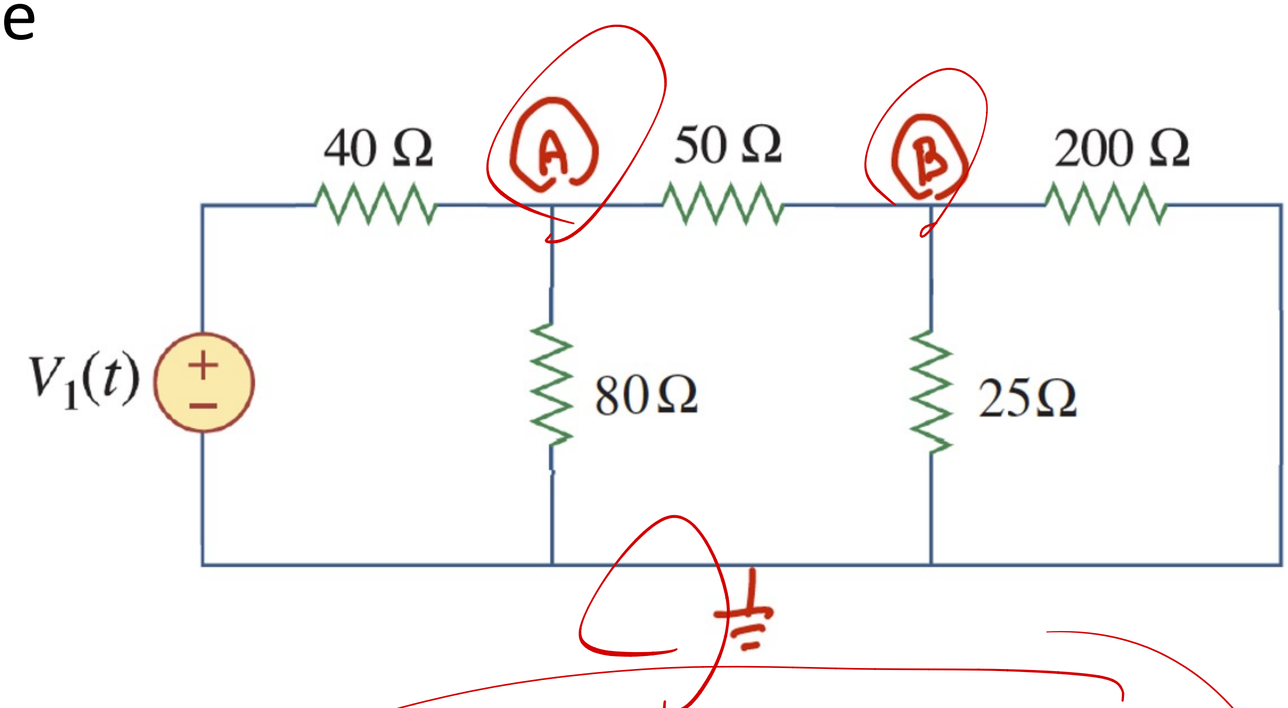
RLC Circuits

- How do we analyze this?



- **Watch** the following (details posted)

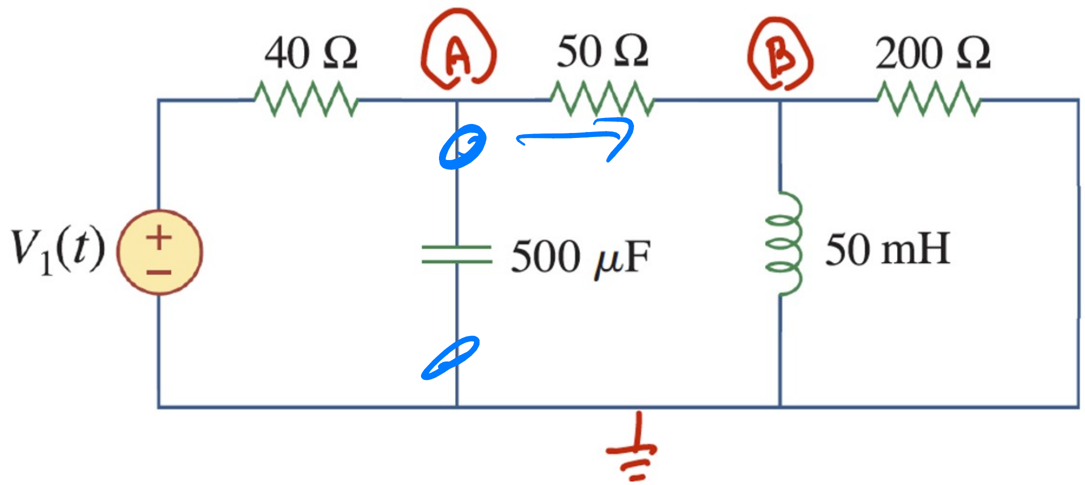
- If only resistors, then node analysis would be the obvious choice



$$\frac{A - V_1(t)}{40} + \frac{A}{80} + \frac{A - B}{50} = 0$$

$$\frac{B}{200} + \frac{B}{25} + \frac{B - A}{50} = 0$$

- Let's try the same using L & C definitions

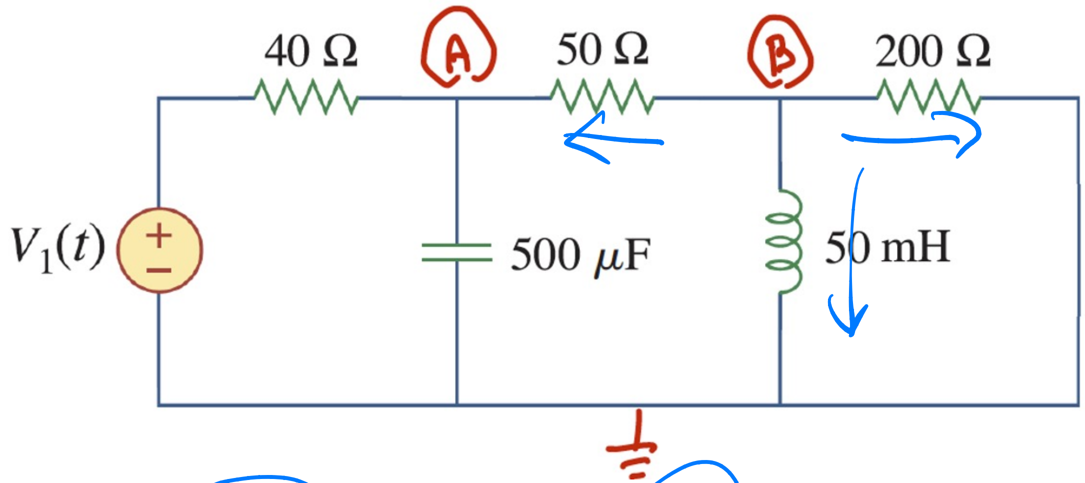


- Node A:

$$\frac{A(t)-V_1(t)}{40}+i_C(t)+\frac{A(t)-B(t)}{50}=0$$

$$0.0005\frac{dA(t)}{dt}$$

$$\frac{dA(t)}{dt}+90\,A(t)-40\,B(t)=50\,V_1(t)$$



- Node B:

$$\frac{B(t)}{200} + i_L(t) + \frac{B(t) - A(t)}{50} = 0$$

$$\frac{1}{L}\int^{t} B(s)\,ds$$ and take a time derivative

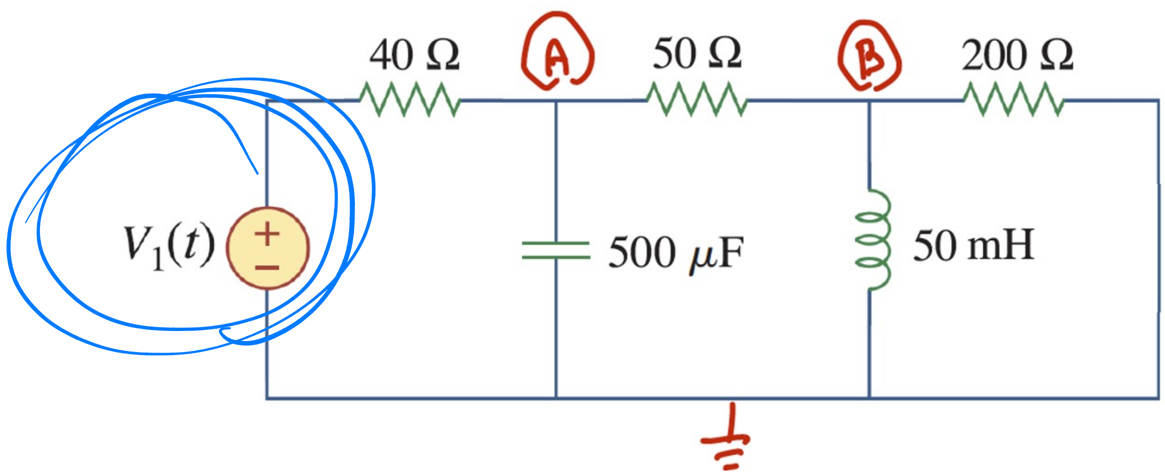
$$\frac{1}{200}\frac{dB(t)}{dt} + \frac{1}{0.05}\,B(t) + \frac{1}{50}\frac{dB(t)}{dt} - \frac{1}{50}\frac{dA(t)}{dt} = 0$$

$$\frac{dB(t)}{dt} + 800\,B(t) - 0.8\,\frac{dA(t)}{dt} = 0$$

- Result is a pair of differential equations

$$\frac{dA(t)}{dt} + 90\,A(t) - 40\,B(t) = 50\,V_1(t)$$

$$\frac{dB(t)}{dt} + 800\,B(t) - 0.8\,\frac{dA(t)}{dt} = 0$$



- How to solve?

$$\frac{dA(t)}{dt} + 90\,A(t) - 40\,B(t) = 50\,V_1(t)$$

$$\frac{dB(t)}{dt} + 800\,B(t) - 0.8\,\frac{dA(t)}{dt} = 0$$

  - Linear ✓
  - Constant coefficient ✓
  - Homogeneous and particular solutions
  - Steps:
    1. Form a single equation from the pair
    2. Find the homogeneous solution
    3. Find the particular solution
    4. Solve for the unknowns

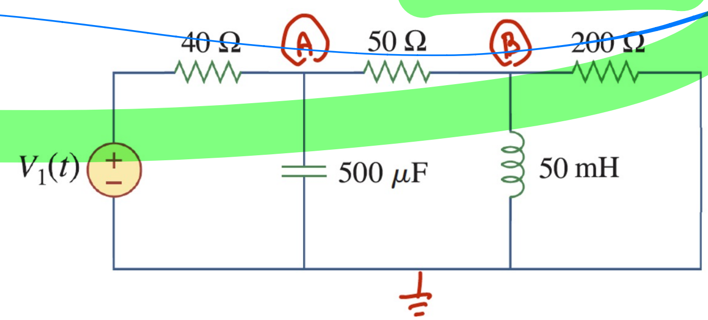
1. Form a single equation from the pair
   – Some magic yields

$$\frac{d^2A(t)}{dt^2} + 858\ \frac{dA(t)}{dt} + 72{,}000\ A(t) = 50\ \frac{dV_1(t)}{dt} + 40{,}000\ V_1(t)$$

and

$$\frac{d^2B(t)}{dt^2} + 858\ \frac{dB(t)}{dt} + 72{,}000\ B(t) = 40\ \frac{dV_1(t)}{dt}$$

   – Continue with $B(t)$



$$\frac{d^2B(t)}{dt^2} + 858\ \frac{dB(t)}{dt} + 72{,}000\ B(t) = 40\ \frac{dV_1(t)}{dt}$$

2. Find the homogeneous solution

$$\frac{d^2B(t)}{dt^2} + 858\ \frac{dB(t)}{dt} + 72{,}000\ B(t) = 0$$

– Characteristic polynomial: $s^2 + 858s + 72{,}000 = 0$

  - Roots, $s = -94.3, -764$

$$B_{homogeneous}(t) = a_1 e^{-94.3t} + a_2 e^{-764t}$$

– Since real parts are negative, then

$$\lim_{t\to\infty} B_{homogeneous}(t) = 0 \quad \Rightarrow \quad \text{“transient”}$$

3. Find the particular solution – depends upon the rhs

– Table lookup:

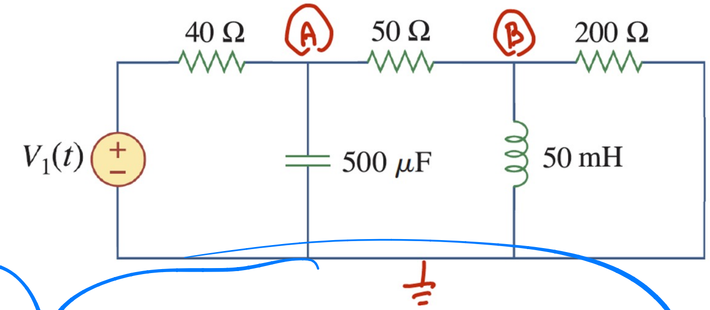
Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$ $= \mathbf{K\cos(\omega t + \theta)}$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

– Focus on sinusoidal sources – why? Let

$$V_1(t) = 10\cos(500t) \text{ volts}$$

4. Find the unknowns K and θ



$$\frac{d^2B(t)}{dt^2} + 858\,\frac{dB(t)}{dt} + 72{,}000\;B(t) = -40{,}000\sin 500t$$

$$B(t) = K\cos(500t + \theta)$$

– Grind through calculus, algebra, and trig

$$B_{steady-state}(t) = 431\cos(500t - 22.5°)\;mV$$

- **Summary** – while it works, the method is tedious and we need want a better approach → the “phasor” method

The Phasor Concept

Extend a sinusoid with an imaginary part

$$x(t) = A\cos(\omega t + \phi)$$

$$x(t) = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$$

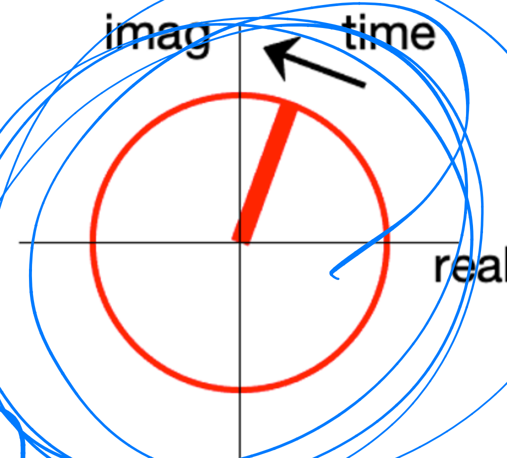
Euler's identity

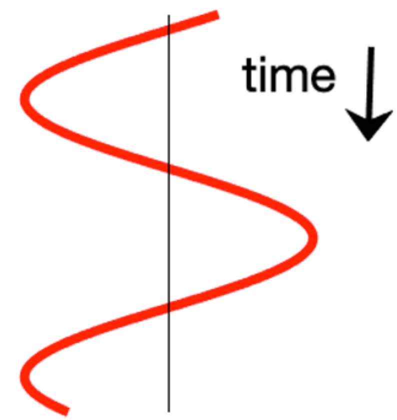
$$= Ae^{j(\omega t + \phi)}$$

$$= \underbrace{Ae^{j\phi}}_{= \mathbf{A}\text{, the "phasor"}} \underbrace{e^{j\omega t}}_{\text{Rotating part}}$$

Bold for complex

$A\cos\phi$
$+ jA\sin\phi$





- Time functions to phasors

$$x(t) = 10\cos\omega t$$

$$\Rightarrow \mathbf{X}e^{j\omega t} = 10\cos\omega t + j10\sin\omega t$$

$$= 10(\cos\omega t + j\sin\omega t)$$

$$= 10\,e^{j\omega t}$$

$$\text{so } \mathbf{X} = 10 = 10\angle 0°$$

10 + j 0

A cos φ

+ j A sin φ

- If $x(t) = 10\cos(500t + 30°)$ find the phasor

$$\mathbf{X}e^{j\omega t} = 10\cos(500t + 30°) + j10\sin(500t + 30°)$$

$$= 10(\cos 500t \cos 30° - \sin 500t \sin 30°)$$
$$+ j10(\cos 500t \sin 30° + \sin 500t \cos 30°)$$

10 cos 30°

+ j 10 sin 30°

$$= 10\cos 500t\,(\cos 30° + j\sin 30°)$$
$$+ j10\sin 500t\,(\cos 30° + j\sin 30°)$$

$$= 10(\cos 30° + j\sin 30°)(\cos 500t + j\sin 500t)$$

$$= \underbrace{(8.66 + j5)}_{\mathbf{X}}\, e^{j500t} = \underbrace{10e^{j30\pi/180}}_{\mathbf{X}}\; e^{j500t}$$

Also, $10\angle 30°$

- If $x(t) = 4\cos(\omega t + 45°)$ then

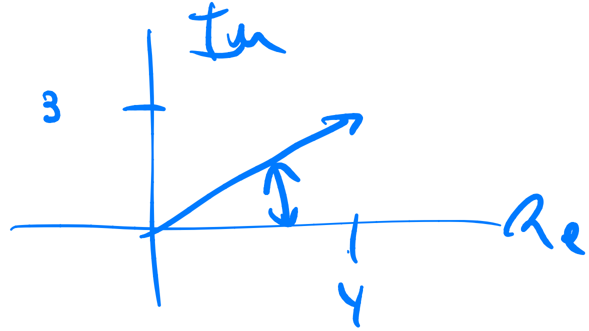
$$\mathbf{X} = 4\angle 45°$$

$$= 4\cos 45° + j4\sin 45°$$

$$= 2\sqrt{2} + j\,2\sqrt{2}$$

$$= 2\sqrt{2}\,(1 + j)$$

And back again



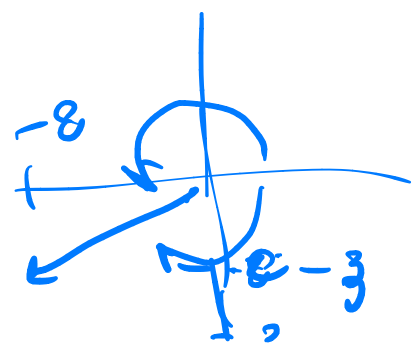
- If $\mathbf{X} = 4 + j3 = \sqrt{4^2 + 3^2}\,\angle \tan^{-1}\frac{3}{4} = 5\angle 36.9°$

$$\text{then } x(t) = 5\cos(\omega t + 36.9°)$$

- If $\mathbf{X} = -8 - j3 = \sqrt{(-8)^2 + (-3)^2}\,\angle \tan^{-1}\frac{-3}{-8}$

$$= \sqrt{73}\,\angle -162°$$

$$\text{then } x(t) = \sqrt{73}\cos(\omega t - 162°)$$



Practice problem: If $x(t) = 100\cos(100t + 102°)$, find $\mathbf{X}$

$-20.6 + j97.8$

Practice problem: If $x(t) = 12\cos(100t + 12°)$, find $\mathbf{X}$

$11.7 + j2.49$

Practice problem: If $x(t) = 12\sin(100t + 12°)$, find $\mathbf{X}$

$2.49 - j211.7$

Practice problem: If $x(t) = 13\cos(100t - 202.6°)$, find $\mathbf{X}$

$-12 + j5$

Practice problem: If $\mathbf{X} = 4 + j2$, find $x(t)$

$4.47\cos(\omega t + 26.6°)$

Practice problem: If $\mathbf{X} = 7.5\angle 31°$, find $x(t)$

$47.5\cos(\omega t + 31°)$