A Quantitative Analysis of the Effect of Block and Pad Size On Perfect Reconstruction Noncausal IIR Digital Filters Using Forward/Backward Block Recursion

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***Abstract* — Two-channel perfect reconstruction filter banks have many applications in communications and signal processing, such as sub-band coding. While implementing a perfect reconstruction filter bank is relatively straightforward using FIR filters, the data must be segmented into blocks and filtered forward and backward if one wishes to use noncausal IIR filters. The concept of block filtering is neither new nor novel, and has been well known for many years. However, a need exists to examine the effects on the error in the perfect reconstruction as related to both the block size and the overlap pad. We present an analysis based on the time constant of the largest magnitude pole of a two-channel, perfect reconstruction 8th order Butterworth filter bank. The choice of an 8th order filter is arbitrary and could just as easily be any order filter. We also examine the sensitivity of the error in perfect reconstruction due to the size of the block as well as the size of the overlap pad. The effects of a quasi-linear implementation of the filter bank are also discussed.**

***Index Terms*—Noncausal IIR filters, perfect reconstruction, forward/backward block filtering.**

I. Introduction

There has been extensive research into the properties of orthogonal FIR and wavelet based perfect reconstruction (PR) filter banks [1]. The study of noncausal IIR filters and their usefulness in PR is somewhat less well known and requires the use of block recursion filtering due to the infinite response of the filtered segment [2]. The low-Q poles of the Butterworth filter result in a fast decay of the transient, implying that the impulse response is negligible after several time constants. By truncating the impulse response, we can effectively utilize a stable backward filter, allowing the filtered blocks to be recombined using the overlap-add method [2] [3].

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For the power-complementary half band PR Butterworth filter bank, the unity gain first order causal lowpass analysis filter is $H(z)=\frac{1}{2}(1+z)$. The corresponding highpass analysis filter is given by $H(-z^{-1})=\frac{1}{2}(1-z^{-1})$ and is anticausal. Similarly, the lowpass synthesis filter given by the anticausal $H(z^{-1})=\frac{1}{2}(1+z^{-1})$ has a complementary highpass synthesis filter given by the causal $H(-z)=\frac{1}{2}(1-z)$. Upon reconstruction, the familiar result

$$H(z)H(z^{-1})+H(-z)H(-z^{-1})=1 \qquad (1)$$

is obtained.

Long data records can be filtered using noncausal IIR filters by block processing. The signal is divided into blocks and each block is filtered in both the forward and backward directions. The filtered block is then combined with the following filtered block using the overlap-add method or the overlap-save method (Fig. 1) and the process is repeated until the end of the time-series recording is reached..

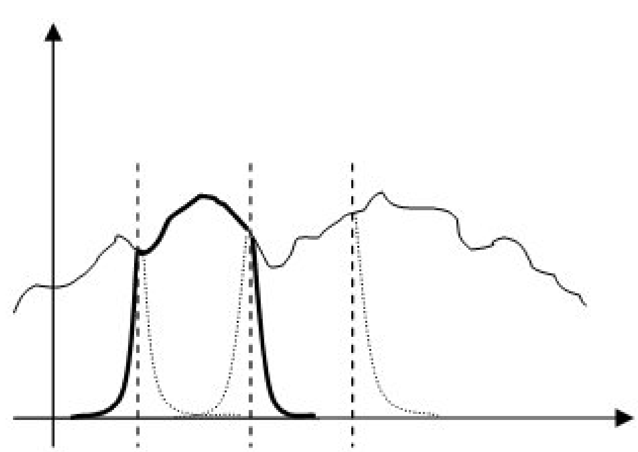

Fig. 1. Given an arbitrary signal, it can be divided into blocks and filtered. The basic idea of the forward/backward recursion filtering is to filter the block forward, creating a forward transient, then time reverse (flip) the filtered segment and filter through the backward filter. Stable backward noncausal filtering is possible if the data record is of finite duration.

II. Time Constants and Block Size

The 8th order Butterworth filter produces a maximum absolute valued pole at $p_{\max} = 0.8207$. As the largest valued pole, it controls the rate of decay of the impulse response. The time constant can be appropriately approximated by

$$\tau = \sum_{n=0}^{\infty} (p_{\max})^n = \frac{1}{1-0.8207} \approx 5.6$$

Rounding to an integer number of samples for the purposes of analysis, $\tau = 6$.

We define the block size as the length (in samples) of the portion of the signal we are filtering. Similarly, we define the pad size as the length (in samples) of the zero-pad appended to the block to hold the transient. The increase in the pad will be incremented in integer multiples of τ. For quantitative analysis, we investigate the decrease of error by increasing both the block size and pad size as a function of τ. Since perfect reconstruction implies no error, the observed error in the reconstruction is due to the truncation of the impulse response and is not related to the design or function of the filter. Clearly, it is impossible to capture the entire impulse response from an IIR filter as it is infinite.

III. Filter Analysis

Consider the following signal obtained from a sharp microelectrode recording from neuron RPD-1 in the parietal ganglion of the central nervous system of the pond snail *Lymnaea stagnalis* [4].

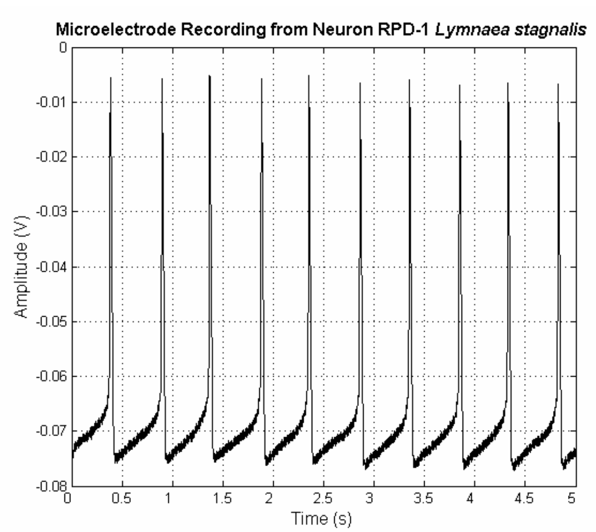

Fig. 2. Waveform generated by neuron RPD-1 in the CNS of the pond snail *Lymnaea stagnalis.*

Using the PR condition defined in (1), the prototype 8th order Butterworth lowpass filter has 8 zeros at -1 and 8 poles on the imaginary axis.

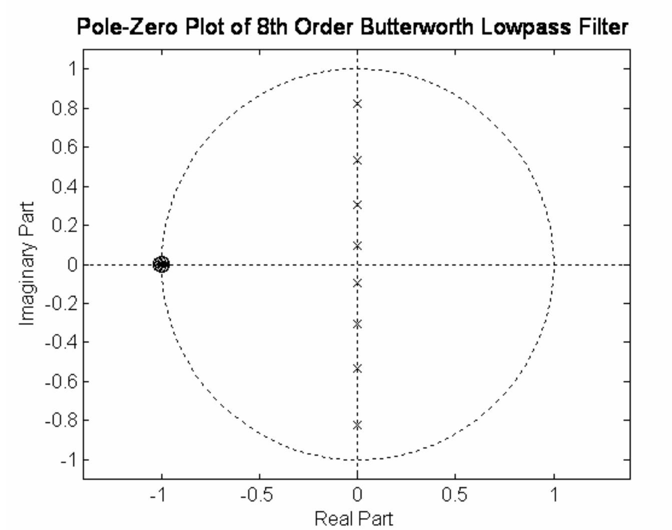

Fig. 3. Pole zero plot of 8th order Butterworth lowpass filter. The cluster of zeros around -1 (not on -1) represents computational error in MATLAB™'s bilinear transform converting the continuous time filter to discrete time.

The highpass version of the prototype filter is trivially obtained by placing the zeros at 1. However, in order to effect a noncausal filter, we must time reverse the block. This can be achieved essentially by flipping the data in the block from left to right. MATLAB™ performs this function by calling the `fliplr` command [5]. The maximally decimated PR filtering sequence then is 1) filter the block forward through the causal lowpass analysis filter *H(z)*, 2) downsample by two, 3) upsample by two, 4) flip the result from left to right (time reverse), 5) filter the result through the noncausal lowpass synthesis filter *H(-z)*, 6) flip the result once again to restore the original orientation. The same procedure is performed on the highpass channel and the two results are summed to reconstruct the original signal. There is, of course, a simple matter of accounting for the factor of 2 generated by the two paths. Some sources account for this by defining the resulting reconstruction as $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2$ [1].

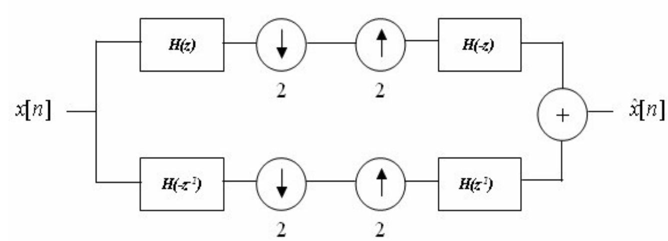

Fig. 4. Diagrammatic representation of a maximally decimated 2 channel filter bank. The conditions imposed on the filters ensure perfect reconstruction.

As a demonstration of the reconstruction, we take the time series generated by neuron RPD-1 shown in figure 2 and examine the error in reconstruction.

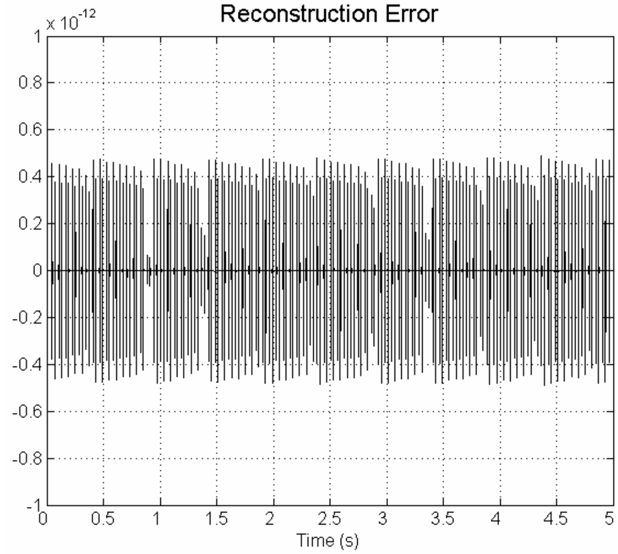

Fig. 5. The reconstruction error for the maximally decimated signal seen in fig 2.

The above figure was generated using a pad size of 20τ and a block size of 20τ. While there is error, at 10^{-13} it is approaching the limit of the machine error. The bursts represent the error in truncation due to the overlap of the tails, as the summation of the trailing tail of one block and the forward tail of the next block overlap and add. We can improve this error by increasing the block size to 40τ.

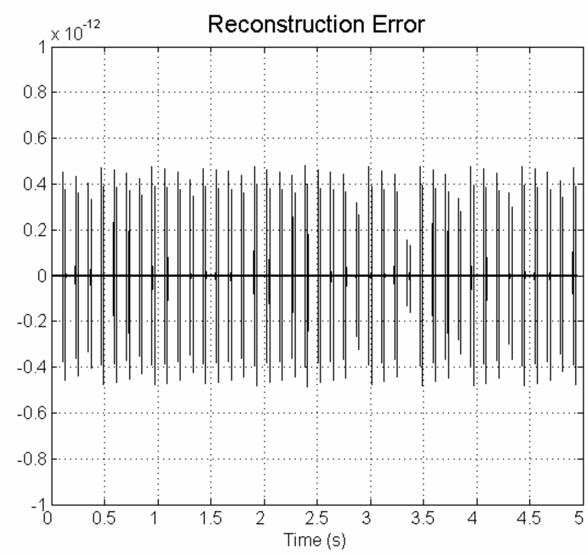

Fig. 6. While the height of the error is only marginally affected, clearly there is less error than with the smaller pad size. This is due to the fact that more of the transient is accounted for and cancelled in the larger pad size. The error that is seen is due to the truncation of the infinite impulse response.

In fact, as the pad size increases from 1 to 20τ and the block size increases from 20 to 40τ, the decrease in error can be seen as a drop of approximately 9dB for every τ increase in the pad length. This is reasonable since $1/e \approx 0.3679$ and $20\log(0.3679) \approx -9\text{dB}$. As expected, as the block size is increased from 20 to 120τ, the error throughout the range of pad sizes decreases. However, the decrease in error due to block size becomes increasing less significant as the block size increases, eventually converging to approximately 0.5dB per 20τ increase in the length of the block.

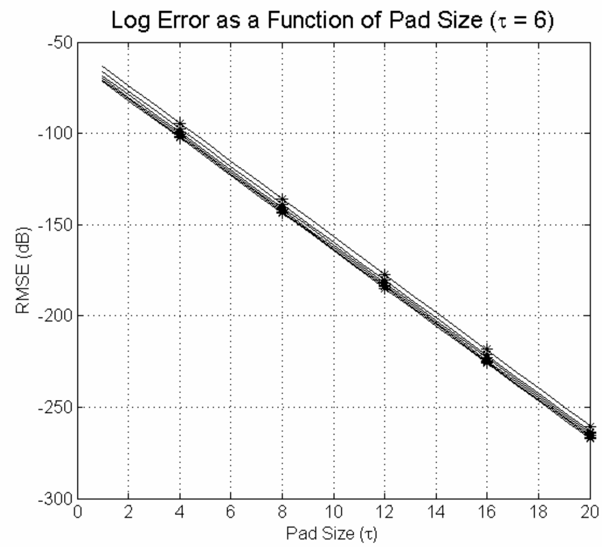

Fig. 7. An analysis of block sizes from 20 to 120τ. The error is converging with increased block size since pad sizes are not correspondingly increasing. This is to be expected since the error is a result of truncation of the transients, which is a function of pad size.

That the error converges is not surprising. The reconstruction is much more sensitive to the overlap pad length since this is where the transients are contained. Since the source of the error is in the truncation of the infinite impulse response, simply increasing the block size without significantly increasing the overlap pad size relative to the increased block size does little to account for the transients. Intuitively, the more of the impulse response that can be compensated for in the overlap pad, the greater the reduction in error in reconstruction.

IV. QUASI-LINEAR PHASE

There is another way to further reduce the reconstruction error, independent of the pad size. The maximally flat Butterworth filter produces a one sided impulse response. That is to say, the impulse response is not symmetric. The analysis and synthesis filters in the Butterworth PR filter bank forms an orthogonal pair. A thorough treatment of this can be found in [2]. The purpose for highlighting this fact is that it is well known that orthogonality destroys symmetry in the impulse response, and symmetry is a necessary, though not sufficient, condition for linear phase. We can, however, form an approximately symmetric impulse response by creating two new filters from each of the analysis and synthesis filters. In the case of the 8^{th} order Butterworth lowpass filter, $H_{lp}(z) = \dfrac{(1+z^{-1})^8}{(1+r^8 z^{-8})}$. The zeros are of course at -1, and all of the poles lie along the imaginary axis. We pair the pole closest to the unit circle with the pole closest to zero to form a new filter. Similarly, we combine the two poles between the inner and outer poles and form another filter. Clearly, if we multiply these two new filters together we generate the original filter. However, if we time reverse the filtered data emerging from the new 4^{th} order filter created from the inner and outer poles before filtering with the new filter created from the two pair of middle poles, we have essentially reduced the effect of all eight poles affecting phase (delay) in the forward filtering process. This is similar to the way MATLAB™ eliminates phase effects by using the `filtfilt` command [5]. In this process, the whole sequence is filtered forward through the filter, introducing phase. Then the whole sequence is filtered backward through the filter, essentially "undoing" the phase distortion.

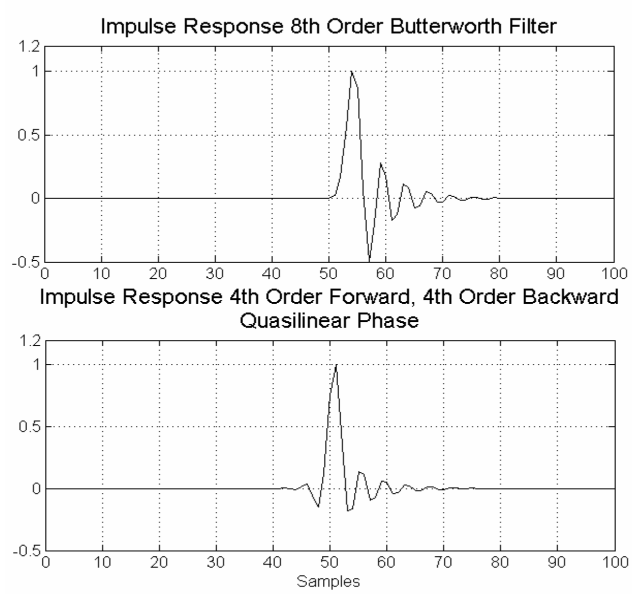

Fig 8. Top. The 8^{th} order Butterworth lowpass filter produces a one-sided impulse response. Bottom. By recombining the poles into two 4^{th} order filters and time reversing the filtered data of the first before filtering with the second, we reduce the effects of phase. Notice the difference in the delay of the impulse response. The impulse was positioned at sample 51.

As one would expect, there is less reconstruction error when there is less phase distortion. To illustrate this, consider the difference in the reconstruction error of an impulse filtered through an unaltered 8^{th} order noncausal IIR Butterworth PR filter bank with an impulse filtered through a quasilinear phase implementation of the same filter bank (fig. 9). As expected, the reconstruction error using the quasilinear phase implementation is less.

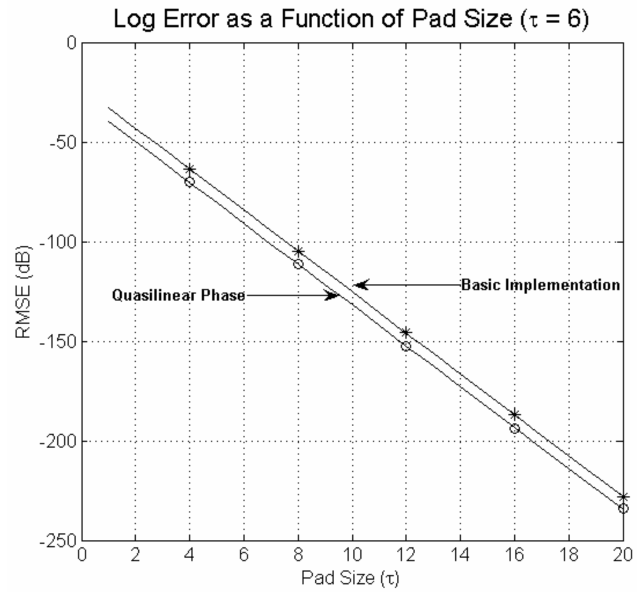

Fig 9. The quasilinear phase implementation is approximate 7dB lower in error than the unaltered Butterworth filter implementation.

V. Conclusion

A quantitative analysis of noncausal IIR filter banks has been presented. It is clear that noncausal IIR filters offer flexibility and alternatives to standard FIR and wavelet implementations of PR filter banks. Forward/backward block recursive filtering provides a means of implementing noncausal IIR PR filter banks. The reconstruction error created using noncausal IIR filters can be reduced significantly by increasing the block and pad sizes. Further reductions can be achieved by adapting the poles to produce a quasilinear phase response.

References

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