Accurate Rectangular Window Subspace Tracking

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Rectangular Window Subspace Tracking

- Given a sequence of column vectors $\mathbf{x}_t$, we can define the $n \times c$ overlapping matrices, M_{old} and $\tilde{M}_{new}$, along with their SVDs, as

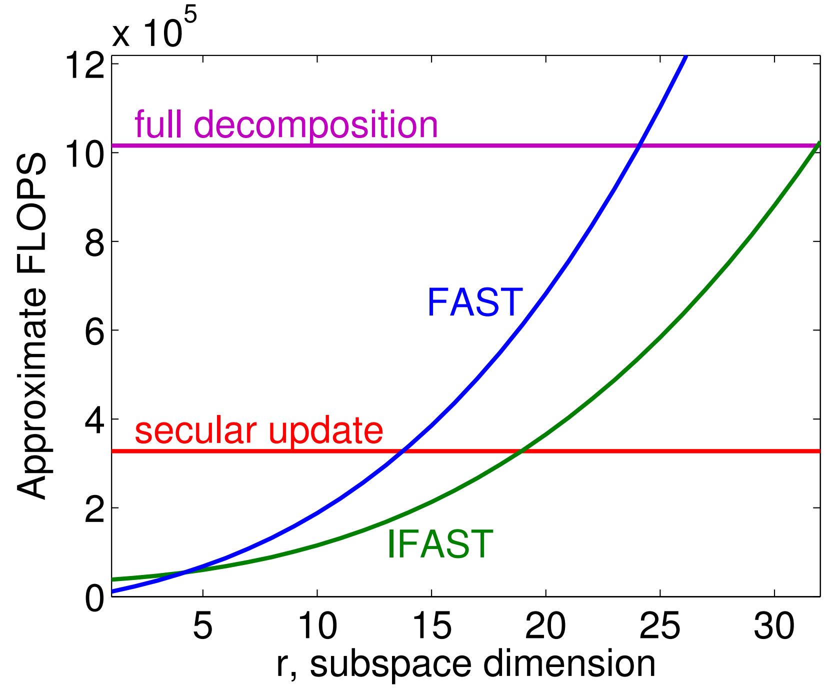
$$\begin{aligned} M_{old} &= [\, \mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \cdots \;\; \mathbf{x}_c \,] &&= U_{old}\Sigma_{old}V_{old}^H \\ \tilde{M}_{new} &= \quad [\, \mathbf{x}_2 \;\; \cdots \;\; \mathbf{x}_c \;\; \mathbf{x}_{c+1} \,] &&= \tilde{U}_{new}\tilde{\Sigma}_{new}\tilde{V}_{new}^H \end{aligned}$$

- Using the rank-two secular function, we can determine the singular values and left singular vectors of $\tilde{M}_{new}$ from the singular values and left singular vectors of M_{old} with one $O(n^3)$ matrix product

- Using the IFAST algorithm, we can determine an approximation to the r largest singular values and left singular vectors of $\tilde{M}_{new}$ from the r largest singular values and left singular vectors of M_{old} with one $O(nr^2)$ and two $O(r^3)$ matrix products

Computation Comparison

- $M_{old}, \tilde{M}_{new} \in \mathbb{C}^{32\times 32}$
- Similar plot for other matrix dimensions.
- The full decomposition and the secular update produce identical results.
- Offset in IFAST at $r = 0$ comes from $O(nc)$ terms.



- For large r, the dominant term in the IFAST computation is an $r+2 \times r+2$ eigendecomposition.
- All methods calculate singular values and left singular vectors only.

Secular Update Method

- If we define the two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} = U_{old}^H \mathbf{x}_1, \qquad \mathbf{b} = U_{old}^H \mathbf{x}_{c+1}$$

then we can define the matrix $\tilde{G}$ as

$$\tilde{G} = U_{old}^H \tilde{M}_{new} \tilde{M}_{new}^H U_{old} = \Sigma_{old}^2 - \mathbf{a}\mathbf{a}^H + \mathbf{b}\mathbf{b}^H$$

which is a diagonal matrix plus two rank one matrices.

- If we write the eigendecomposition of $\tilde{G}$ as

$$\tilde{G} = \tilde{U}_G \tilde{\Sigma}_G \tilde{U}_G^H$$

then the singular values and left singular vectors of $\tilde{M}_{new}$ are

$$\tilde{\Sigma}_{new} = \sqrt{\tilde{\Sigma}_G} \qquad \text{and} \qquad \tilde{U}_{new} = U_{old} \tilde{U}_G$$

The Eigendecomposition of $\tilde{G}$

- The eigenvalues of $\tilde{G}$ are the roots of the rank-two secular equation

$$w(\lambda) = \left(1 - \sum_{j=1}^{n} \frac{|a_j|^2}{\sigma_j^2 - \lambda}\right)\left(1 + \sum_{j=1}^{n} \frac{|b_j|^2}{\sigma_j^2 - \lambda}\right) + \left|\sum_{j=1}^{n} \frac{a_j^* b_j}{\sigma_j^2 - \lambda}\right|^2$$

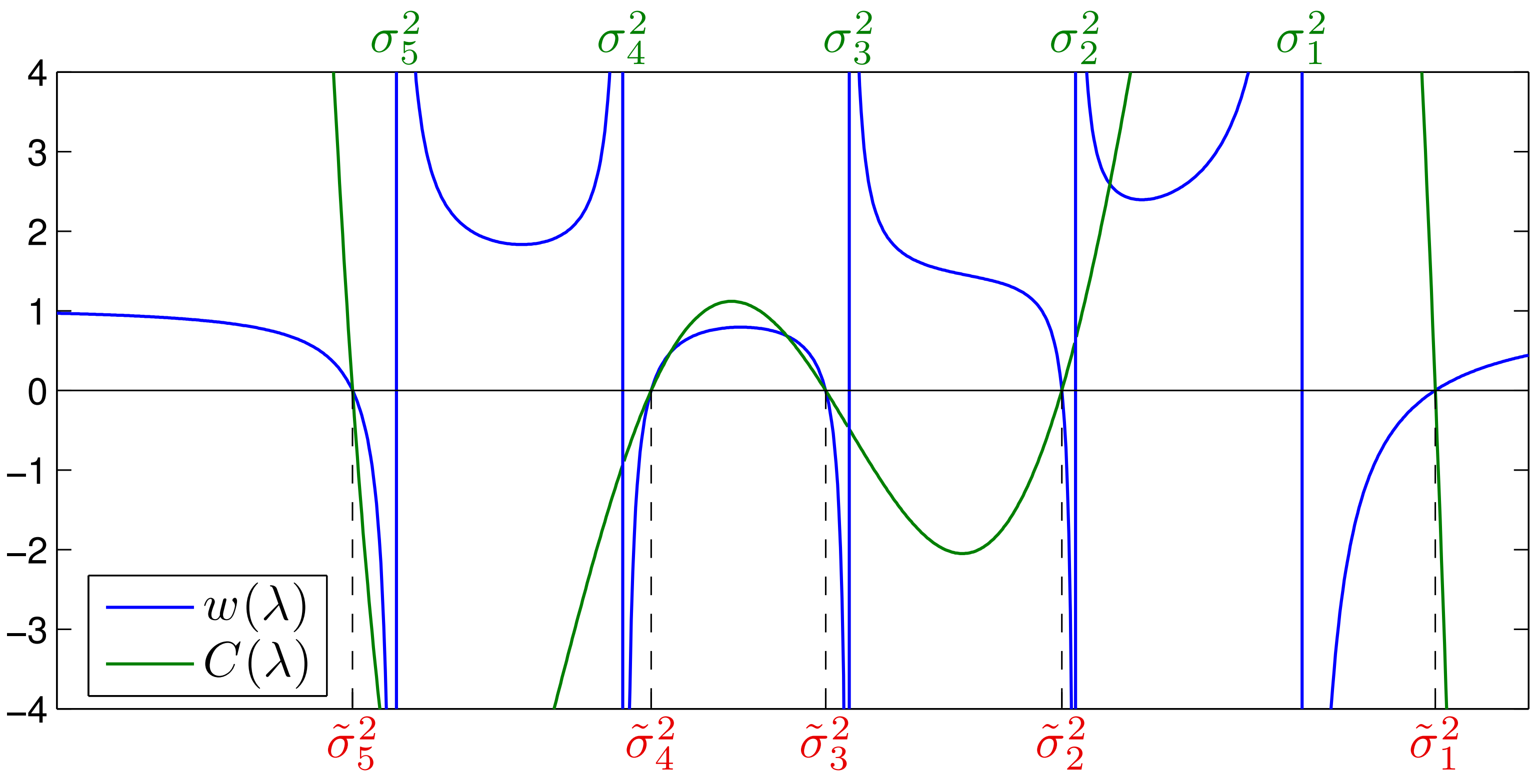
$$w(\lambda) = w_a(\lambda)\, w_b(\lambda) + |w_{ab}(\lambda)|^2$$

where $w_a(\lambda)$ is the secular function for $\Sigma_{old}^2 - \mathbf{a}\mathbf{a}^H$,
and $w_b(\lambda)$ is the secular function for $\Sigma_{old}^2 + \mathbf{b}\mathbf{b}^H$

- The unnormalized ith eigenvector of $\tilde{G}$ is

$$\frac{\tilde{\mathbf{u}}_i}{c_s} = \left(\Sigma_{old}^2 - \tilde{\sigma}_i^2 I\right)^{-1}\left(\mathbf{a} + \frac{w_a(\tilde{\sigma}_i^2)}{w_{ab}(\tilde{\sigma}_i^2)}\,\mathbf{b}\right)$$

Rank-Two Secular Function



Poles of $w(\lambda)$ are squares of singular values of M_{old}

Roots of $w(\lambda)$ and $C(\lambda)$ are squares of singular values of $\tilde{M}_{new}$

Note that each new eigenvalue is bounded by $\sigma_{i+1}^2 < \tilde{\sigma}_i^2 < \sigma_{i-1}^2$

The IFAST Approximation

- M_{old} shares all but one column with $\tilde{M}_{new}$, therefore U'_{old} is a reasonable approximation to the columnspace of $\tilde{U}'_{new}$, except for the contribution of $\mathbf{x}_1$ and $\mathbf{x}_{c+1}$

- If we define $\mathbf{q}_1$ and $\mathbf{q}_2$ as a Gram-Schmidt augmentation to U'_{old},

$$\mathbf{z}_1 = (I - U'_{old}{U'_{old}}^H)\mathbf{x}_{c+1}, \qquad \mathbf{q}_1 = \mathbf{z}_1/\|\mathbf{z}_1\|,$$
$$\mathbf{z}_2 = (I - [\, U'_{old} \,|\, \mathbf{q}_1 \,][\, U'_{old} \,|\, \mathbf{q}_1 \,]^H)\mathbf{x}_1, \qquad \mathbf{q}_2 = \mathbf{z}_2/\|\mathbf{z}_2\|,$$

  then we can define

$$\tilde{M}'_{new} = [\, U'_{old} \,|\, Q \,][\, U'_{old} \,|\, Q \,]^H \tilde{M}_{new}$$

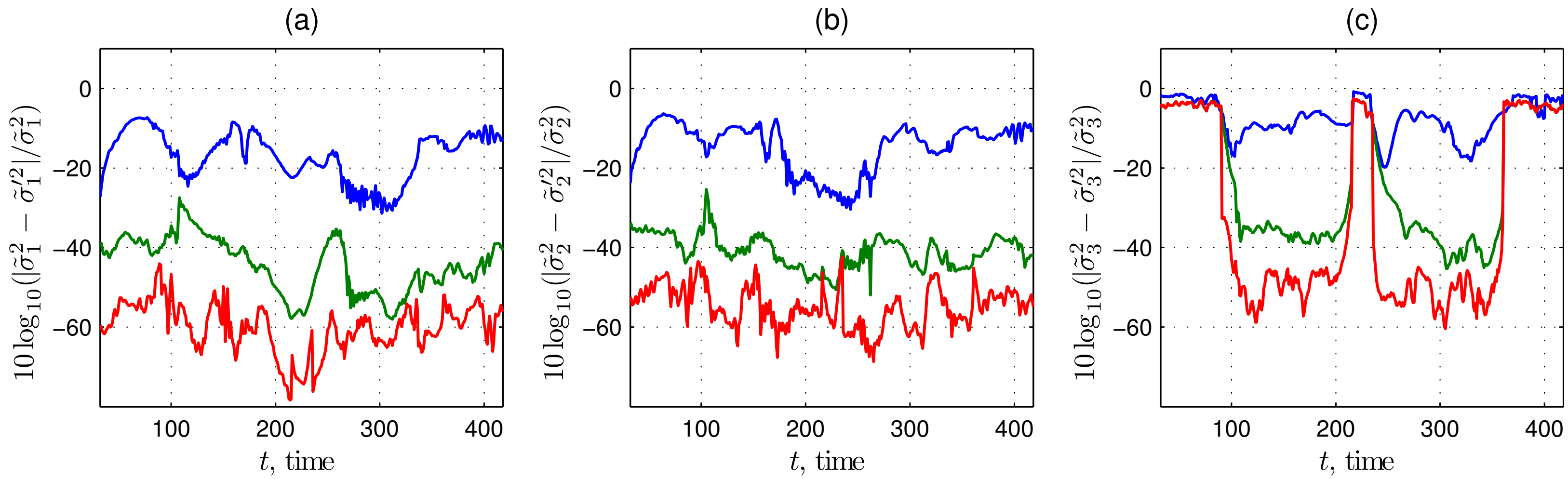
  The r largest singular values and left singular vectors of the rank $r+2$ matrix $\tilde{M}'_{new}$, are reasonable approximations to the r largest singular values and left singular vectors of $\tilde{M}_{new}$

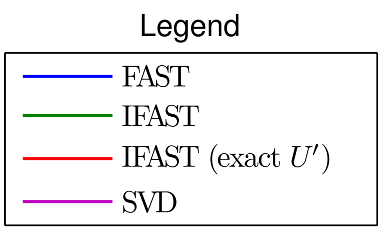
- Substituting $\tilde{M}'_{new}$ for $\tilde{M}_{new}$ is the only approximation in IFAST

The IFAST Algorithm

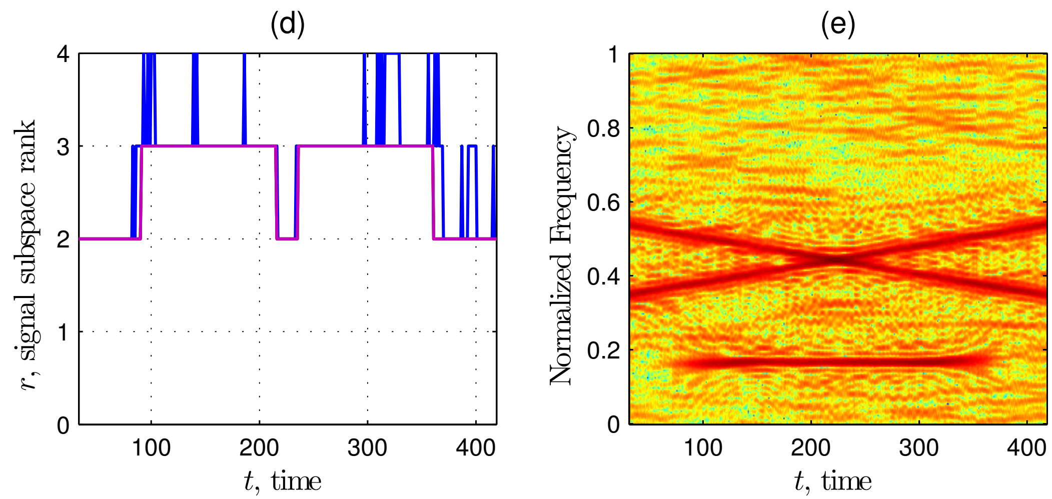
Step		Description
1)	$\mathbf{z}_1 = \left(I - U'_{old}{U'_{old}}^H\right)\mathbf{x}_{c+1}$ $\mathbf{q}_1 = \mathbf{z}_1 / \lVert\mathbf{z}_1\rVert$ $\mathbf{z}_2 = \left(I - [\,U'_{old} \mid \mathbf{q}_1\,][\,U'_{old} \mid \mathbf{q}_1\,]^H\right)\mathbf{x}_1$ $\mathbf{q}_2 = \mathbf{z}_2 / \lVert\mathbf{z}_2\rVert$	Gram-Schmidt augment U'_{old} with the column we are discarding, $\mathbf{x}_1$, and the column we are adding, $\mathbf{x}_{c+1}$, to create the matrix $Q = [\mathbf{q}_1 \ \mathbf{q}_2]$, where $[\,U'_{old} \mid Q\,]^H[\,U'_{old} \mid Q\,] = I$
2)	$\tilde{D} = \Sigma'^2_{old} - U'^H_{old}\mathbf{x}_1\mathbf{x}_1^H U'_{old} + U'^H_{old}\mathbf{x}_{c+1}\mathbf{x}_{c+1}^H U'_{old}$ $\tilde{F} = \left[\begin{array}{c\|c} \tilde{D} & U'^H_{old}\tilde{M}_{new}\tilde{M}_{new}^H Q \\ \hline Q^H\tilde{M}_{new}\tilde{M}_{new}^H U'_{old} & Q^H\tilde{M}_{new}\tilde{M}_{new}^H Q \end{array}\right]$	Create $\tilde{F}$, whose eigendecomposition will give us the SVD of $\tilde{M}'_{new}$. Equivalent to $\tilde{F} = [\,U'_{old} \mid Q\,]^H\tilde{M}_{new}\tilde{M}_{new}^H[\,U'_{old} \mid Q\,]$
3)	$\tilde{U}_f\tilde{\Sigma}_f\tilde{U}_f^H = \tilde{F}$	Take eigendecomposition of $\tilde{F}$
4)	$\tilde{U}'_{new} = [\,U'_{old} \mid Q\,]\tilde{U}_f$ $\tilde{\Sigma}'^2_{new} = \tilde{\Sigma}_f$	Determine the singular values and left singular vectors of $\tilde{M}'_{new}$

IFAST Performance





Sample: t=200, $\tilde{\sigma}_1'^2$	
FAST:	1210.752108327795
IFAST:	1300.793378303474
IFAST (exact U'):	1300.915147265195
SVD:	1300.916314593100



Analytical Analysis of IFAST

- Given U_{old} and Σ_{old} from M_{old}, and defining like before

$$\mathbf{a} = U_{old}^H \mathbf{x}_1, \qquad \mathbf{b} = U_{old}^H \mathbf{x}_{c+1}$$

the matrix $\tilde{G} = U_{old}^H \tilde{M}_{new} \tilde{M}_{new}^H U_{old}$, can be written as

$$\tilde{G} = \Sigma_{old}^2 - \mathbf{a}\mathbf{a}^H + \mathbf{b}\mathbf{b}^H$$

- Given U'_{old} and Σ'_{old} from M_{old}, and defining

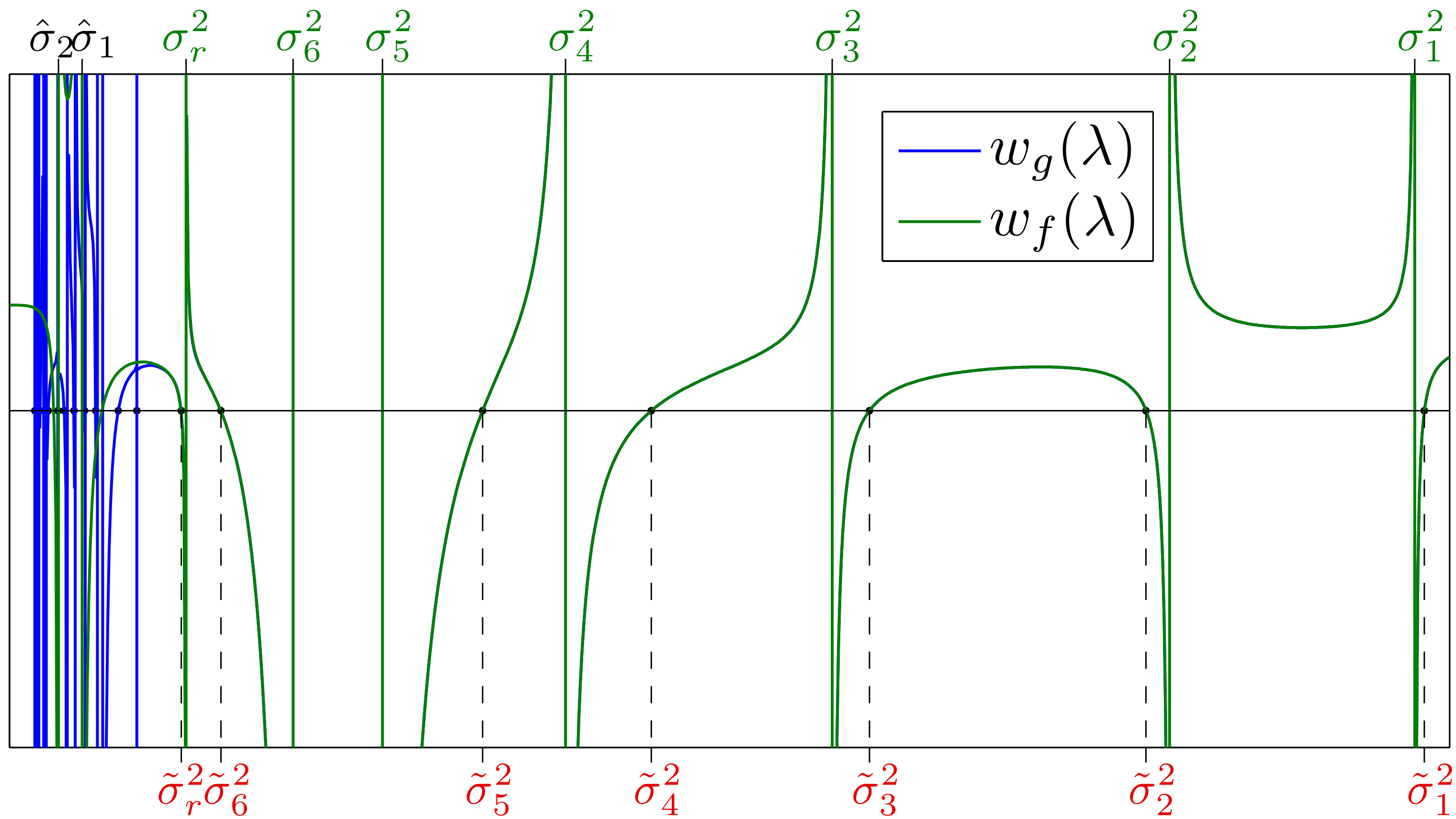
$$\hat{\Sigma} = Q^H M_{old} M_{old}{}^H Q, \qquad \hat{\mathbf{a}} = Q^H \mathbf{x}_1, \qquad \hat{\mathbf{b}} = Q^H \mathbf{x}_{c+1}$$

the matrix $\tilde{F} = [U'_{old}\ Q]^H \tilde{M}_{new} \tilde{M}_{new}^H [U'_{old}\ Q]$, can be written as

$$\tilde{F} = \left[\begin{array}{c|c} \Sigma'^2_{old} & 0 \\ \hline 0 & \hat{\Sigma} \end{array}\right] - \left[\begin{array}{c} \mathbf{a}' \\ \hline \hat{\mathbf{a}} \end{array}\right] \left[\begin{array}{c} \mathbf{a}' \\ \hline \hat{\mathbf{a}} \end{array}\right]^H + \left[\begin{array}{c} \mathbf{b}' \\ \hline \hat{\mathbf{b}} \end{array}\right] \left[\begin{array}{c} \mathbf{b}' \\ \hline \hat{\mathbf{b}} \end{array}\right]^H$$

- We can now use their rank-two secular functions for comparison

Comparison of Secular Functions



Poles of $w_g(\lambda)$ are squares of singular values of M_{old}

Roots of $w_g(\lambda)$ are squares of the singular values of $\tilde{M}_{new}$

Roots of $w_f(\lambda)$ are squares of the singular values of $\tilde{M}'_{new}$

Sequential Rank One Updates

- The rank-two secular function is the product of the two rank-one secular functions plus a cross term, $w(\lambda) = w_a(\lambda)w_b(\lambda) + |w_{ab}(\lambda)|^2$

- We can define the difference of the rank one secular function of $\tilde{G}_b = \Sigma_{old}^2 + \mathbf{b}\mathbf{b}^H$ and $\tilde{F}_b$ as

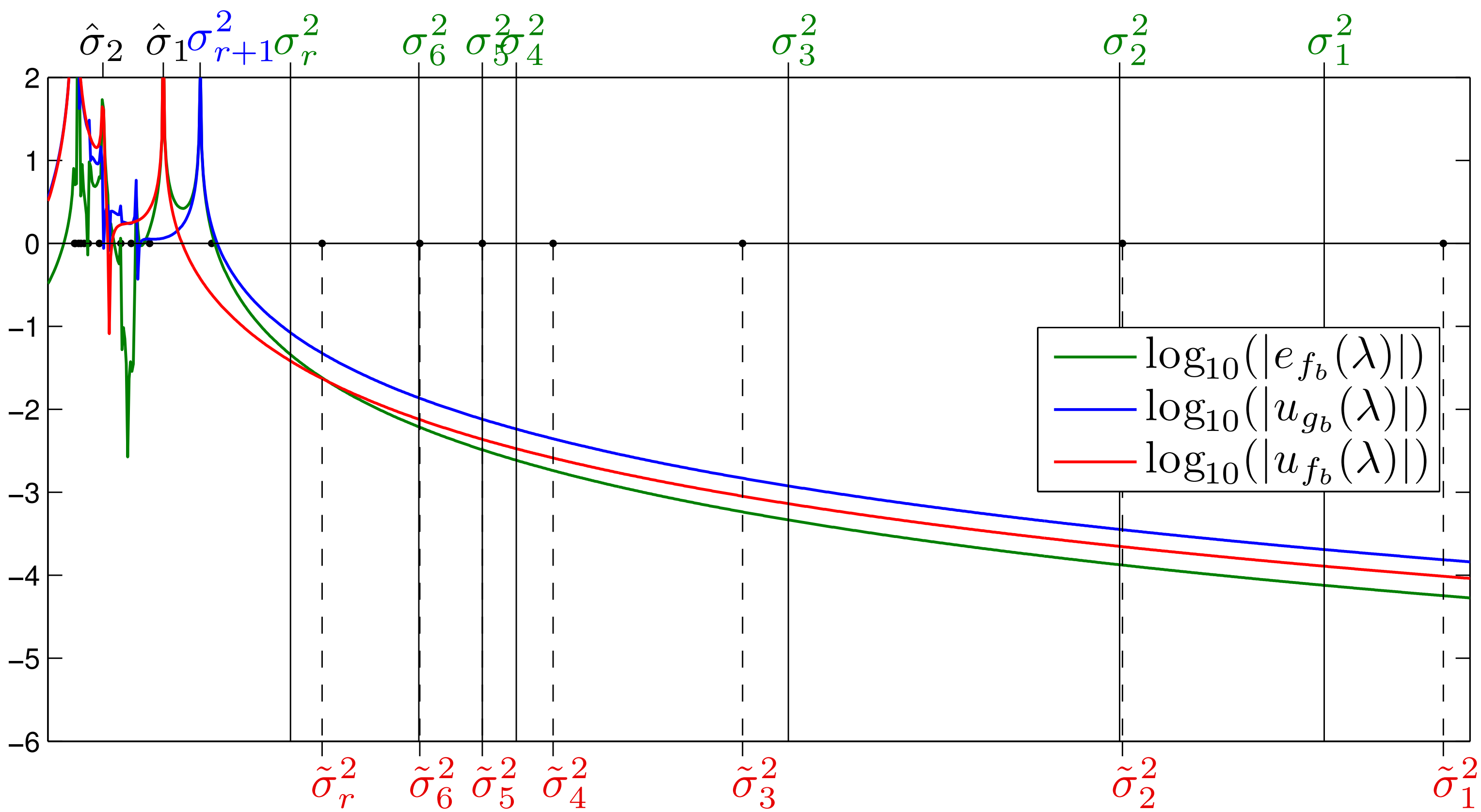
$$e_{f_b}(\lambda) = w_{f_b}(\lambda) - w_{g_b}(\lambda)$$

and the unique parts of $w_{f_b}(\lambda)$ and $w_{g_b}(\lambda)$ as

$$u_{g_b}(\lambda) = -\sum_{j=r+1}^{n} \frac{|b_j|^2}{\lambda} \sum_{i=2}^{\infty} \left(\frac{\sigma_j^2}{\lambda}\right)^i, \qquad u_{f_b}(\lambda) = -\sum_{j=1}^{2} \frac{|\hat{b}_j|^2}{\lambda} \sum_{i=2}^{\infty} \left(\frac{\hat{\sigma}_j}{\lambda}\right)^i$$

- The three functions $e_{f_b}(\lambda)$, $u_{g_b}(\lambda)$, and $u_{f_b}(\lambda)$ are all about the same magnitude in the region of λ that we are interested in, therefore $u_{f_b}(\lambda)$ can be used to give an idea of the error in the approximation

Secular Function Differences



Error and difference in the secular functions for $\tilde{F}_b$ and $\tilde{G}_b$

Drops off rapidly in region on interest, $\lambda > \sigma_r^2$

Summary of Presentation

- IFAST Algorithm - Singular value and left singular vector estimates in $O(nr^2)$ time. Improved the computational efficiency, accuracy, and flexibility of the FAST algorithm.

- Secular Method - Method to find eigendecomposition of a diagonal matrix plus two rank one matrices in $O(n^2)$ time, and update the singular values and left singular vectors of a matrix where one column changes with a single $O(n^3)$ matrix product.

- IFAST Analysis - Show how to use rank-two secular function to analyze accuracy of IFAST algorithm. This gives some insight into where IFAST is applicable.