Efficient and Accurate Rectangular Window Subspace Tracking

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OVERVIEW

In sensor array signal processing, one often needs the principal left singular vectors and singular values of a sequence of array snapshots, or equivalently, the principal eigenvectors and eigenvalues of the sample covariance matrix of those same snapshots.

Here we describe an algorithm which generates approximations to the r principal singular values and left singular vectors of a series of overlapping matrices. Other than two r-dimensional matrix rotations of the singular vectors (one $O(nr^2)$ and one $O(r^3)$), all computation consists of matrix-vector products (ie. $O(nr)$) or equivalent. This algorithm is based on an approximate secular function, which accurately represents the exact secular function in the range of the principal singular values. Details are shown in [1], [2]. (A secular function is related to a characteristic polynomial, and has roots which correspond to the singular values.)

DESCRIPTION OF PROBLEM

Given two overlapping $n \times c$ complex matrices, which might be created from c sequential snapshots from an n element sensor array,

$$M_{old} = \left[\, \mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \mathbf{x}_3 \,|\, \cdots \,|\, \mathbf{x}_{c-1} \,|\, \mathbf{x}_c \,\right],$$
$$M_{new} = \left[\, \mathbf{x}_2 \,|\, \mathbf{x}_3 \,|\, \cdots \,|\, \mathbf{x}_{c-1} \,|\, \mathbf{x}_c \,|\, \mathbf{x}_{c+1} \,\right],$$

along with approximations to:

- Σ'_{old} - the r principal singular values of M_{old}
- U'_{old} - the r principal left singular vectors of M_{old}

we would like to efficiently determine approximations to the r or $r+1$ principal singular values and left singular vectors of M_{new}.

If we create the $n \times 2$ matrix $\dot{Q}$ by Gram-Schmidt augmenting U'_{old} with the two columns that differ between M_{old} and M_{new}, then define the rank $r+2$ matrix

$$M'_{new} = [U'_{old} \,|\, \dot{Q}][U'_{old} \,|\, \dot{Q}]^H M_{new},$$

the r principal singular values and left singular vectors of M'_{new} are good approximations to those of the matrix M_{new}. See [1], [2] for a secular function analysis of the accuracy of this approximation.

Here we present the *Improved Secular Fast Adaptive Subspace Tracking* (ISFAST) algorithm, which determines the exact singular values and left singular vectors of M'_{new} in a computationally efficient manner. Approximating M_{new} by M'_{new} is the **only approximation** in this algorithm.

SUMMARY

We describe the ISFAST algorithm, which is an $O(nr^2)$ rectangular window subspace tracking algorithm that gives approximations to the full set of principal singular values and left singular vectors. Rectangular data windows allow inclusion of a finite set of approximately stationary data, even when the signal subspace is changing rapidly.

The ISFAST algorithm is more accurate and more computationally efficient than the FAST algorithm of Real, Tufts, and Cooley, which is more accurate than the PASTd algorithm and Prony-Lanczos method.

STEPS OF THE ISFAST ALGORITHM

$$\dot{\mathbf{q}}_1 = \frac{\left(I - U'_{old}U'^H_{old}\right)\mathbf{x}_{c+1}}{\left\|\left(I - U'_{old}U'^H_{old}\right)\mathbf{x}_{c+1}\right\|}$$
$$\dot{\mathbf{q}}_2 = \frac{\left(I - U'_{old}U'^H_{old} - \dot{\mathbf{q}}_1\dot{\mathbf{q}}_1^H\right)\mathbf{x}_1}{\left\|\left(I - U'_{old}U'^H_{old} - \dot{\mathbf{q}}_1\dot{\mathbf{q}}_1^H\right)\mathbf{x}_1\right\|}$$
$$\dot{Q} = [\,\dot{\mathbf{q}}_1 \,|\, \dot{\mathbf{q}}_2\,]$$

Create the $n \times 2$ matrix $\dot{Q}$, by Gram-Schmidt augmenting U'_{old} with the column we added and the column we removed from M_{old}, to construct M_{new}.

Note that $[U'_{old}\ \dot{Q}]^H[U'_{old}\ \dot{Q}]^H = I$

$$\hat{T} = \dot{Q}^H M_{old} M_{old}^H \dot{Q}$$
$$\hat{U}\hat{\Sigma}\hat{U}^H = \hat{T}$$
$$Q = \dot{Q}\hat{U}$$

Create the $n \times 2$ matrix Q, by rotating the matrix $\dot{Q}$, such that the 2×2 matrix $\hat{\Sigma} = Q^H M_{old} M_{old}^H Q$ is diagonal

$$Z = U'^H_{old} M_{old} M_{old}^H Q$$
$$\dot{G} = \left[\begin{array}{c|c} \Sigma'^2_{old} & 0 \\ \hline 0 & \hat{\Sigma} \end{array}\right] + \left[\begin{array}{c|c} 0 & Z \\ \hline Z^H & 0 \end{array}\right]$$
$$\dot{U}\dot{\Sigma}\dot{U}^H = \dot{G}$$

Construct the $(r+2) \times 2$ matrix Z, then calculate the eigendecomposition of $\dot{G}$

$$\dot{\mathbf{a}} = \dot{U}^H[\,U'_{old} \,|\, Q\,]^H \mathbf{x}_1$$
$$\dot{\mathbf{b}} = \dot{U}^H[\,U'_{old} \,|\, Q\,]^H \mathbf{x}_{c+1}$$
$$\ddot{H} = \dot{\Sigma} - \dot{\mathbf{a}}\dot{\mathbf{a}}^H + \dot{\mathbf{b}}\dot{\mathbf{b}}^H$$
$$\ddot{U}\ddot{\Sigma}\ddot{U}^H = \ddot{H}$$

Construct the length $r+2$ vectors $\dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$, then calculate the eigendecomposition of $\ddot{H}$

$$\Sigma'^2_{new} = \ddot{\Sigma}$$
$$U'_{new} = [\,U'_{old} \,|\, Q\,]\dot{U}\ddot{U}$$

Calculate the singular values and left singular vectors of M'_{new}

The **construction of** U'_{new} is the **only** $O(\cdot^3)$ matrix-matrix product; all other steps contain only $O(\cdot^2)$ matrix-vector products or equivalent

theoretically equivalent steps

EQUIVALENT REFERENCE ALGORITHM

Construct $\dot{Q}$ as done in the first three steps above

$$F = [\,U'_{old} \,|\, \dot{Q}\,]^H M_{new} M_{new}^H [\,U'_{old} \,|\, \dot{Q}\,]$$
$$U_f \Sigma_f U_f^H = F$$

Construct the $(r+2) \times (r+2)$ matrix F, then calculate its EVD

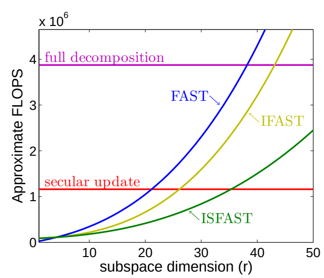
$$\Sigma'^2_{new} = \Sigma_f$$
$$U'_{new} = [\,U'_{old} \,|\, \dot{Q}\,]U_f$$

Calculate the singular values and left singular vectors of M'_{new}

EFFICIENCY EXAMPLE

Computational cost for a 50×50 complex matrix vs. principal subspace dimension, r.

The full decomposition and the secular update determine the full set of exact U_{new} and Σ_{new}, while the other three approximate the r dimensional set of U'_{new} and Σ'_{new}. The "reference algorithm" would cost about the same as FAST.



EIGENDECOMPOSITION OF A SYMMETRIC, DOUBLY BORDERED DIAGONAL MATRIX IS AN $O(r^2)$ OPERATION

The eigenvalues of $\dot{G} = \left[\begin{smallmatrix} \Sigma'^2_{old} & 0 \\ 0 & \hat{\Sigma} \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & Z \\ Z^H & 0 \end{smallmatrix}\right]$ are the roots of the secular function $\dot{w}(\lambda)$, and are bounded by the singular values of M_{old}, (ie. σ'^2_i)

$$\dot{w}(\lambda) = \Bigg(\underbrace{\hat{\sigma}_1 - \lambda - \sum_{j=1}^{r}\frac{|z_{j,1}|^2}{\sigma'^2_j - \lambda}}_{\dot{w}_1(\lambda)}\Bigg)\Bigg(\underbrace{\hat{\sigma}_2 - \lambda - \sum_{j=1}^{r}\frac{|z_{j,2}|^2}{\sigma'^2_j - \lambda}}_{\dot{w}_2(\lambda)}\Bigg) - \Bigg|\underbrace{\sum_{j=1}^{r}\frac{z^*_{j,1}z_{j,2}}{\sigma'^2_j - \lambda}}_{\dot{w}_x(\lambda)}\Bigg|^2$$

The ith unnormalized eigenvector of $\dot{G}$ is the linear combination of two vectors times a diagonal matrix, and can be computed in $O(r)$ time

$$\dot{\mathbf{v}}_i = \left[\begin{array}{c|c} \Sigma'^2 - \dot{\sigma}_i I & 0 \\ \hline 0 & I \end{array}\right]^{-1}\left(\begin{bmatrix} \mathbf{z}_1 \\ -1 \\ 0 \end{bmatrix} + \frac{\dot{w}_1(\dot{\sigma}_i)}{\dot{w}_x(\dot{\sigma}_i)}\begin{bmatrix} \mathbf{z}_2 \\ 0 \\ -1 \end{bmatrix}\right) \quad \Longrightarrow \quad \dot{\mathbf{u}}_i = \frac{\dot{\mathbf{v}}_i}{\|\dot{\mathbf{v}}_i\|}$$

EIGENDECOMPOSITION OF A SYMMETRIC, RANK-TWO MODIFICATION OF A DIAGONAL MATRIX IS AN $O(r^2)$ OPERATION

The eigenvalues of $\ddot{H} = \dot{\Sigma} - \dot{\mathbf{a}}\dot{\mathbf{a}}^H + \dot{\mathbf{b}}\dot{\mathbf{b}}^H$ are the roots of the secular function $\ddot{w}(\lambda)$, and are bounded by the eigenvalues of $\dot{G}$, (which are $\dot{\sigma}_j$)

$$\ddot{w}(\lambda) = \Bigg(\underbrace{1 - \sum_{j=1}^{r}\frac{|\dot{a}_j|^2}{\dot{\sigma}_j - \lambda}}_{\ddot{w}_a(\lambda)}\Bigg)\Bigg(\underbrace{1 + \sum_{j=1}^{r}\frac{|\dot{b}_j|^2}{\dot{\sigma}_j - \lambda}}_{\ddot{w}_b(\lambda)}\Bigg) + \Bigg|\underbrace{\sum_{j=1}^{r}\frac{\dot{a}^*_j\dot{b}_j}{\dot{\sigma}_j - \lambda}}_{\ddot{w}_{ab}(\lambda)}\Bigg|^2$$

The ith unnormalized eigenvector of $\ddot{H}$ is the linear combination of two vectors times a diagonal matrix, and can be computed in $O(r)$ time

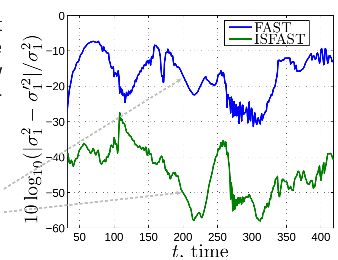
$$\ddot{\mathbf{v}}_i = \left(\dot{\Sigma} - \ddot{\sigma}_i I\right)^{-1}\left(\dot{\mathbf{a}} + \frac{\ddot{w}_a(\ddot{\sigma}_i)}{\ddot{w}_{ab}(\ddot{\sigma}_i)}\dot{\mathbf{b}}\right) \quad \Longrightarrow \quad \ddot{\mathbf{u}}_i = \frac{\ddot{\mathbf{v}}_i}{\|\ddot{\mathbf{v}}_i\|}$$

ACCURACY EXAMPLE

Normalized error of the largest singular value of two chirps. The ISFAST algorithm is usually a few orders of magnitude more accurate than the FAST algorithm.

Values at time $t = 200$

FAST:	$\sigma'^2_1 = 1210.752108327795$
ISFAST:	$\sigma'^2_1 = 1300.793378303474$
SVD:	$\sigma^2_1 = 1300.916314593100$



REFERENCES

[1] T. M. Toolan, "Advances in sliding window subspace tracking," Ph.D. dissertation, University of Rhode Island, 2005.

[2] T. M. Toolan and D. W. Tufts, "Improved fast adaptive subspace tracking," in *Proc. Thirteenth Annual Adaptive Sensor Array Processing Workshop (ASAP)*, 2005.