Improved Fast Adaptive Subspace Tracking

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Sliding Rectangular Window

- Given a sequence of length n column vectors $\boldsymbol{x}_t$, for $t = 1, 2, \cdots$, we can define the $n \times c$ matrices M and $\tilde{M}$ as

$$\begin{aligned} M &= [\, \boldsymbol{x}_{t-c} \;\; \boldsymbol{x}_{t-c+1} \;\; \cdots \;\; \boldsymbol{x}_{t-1} \,] \\ \tilde{M} &= \qquad\quad [\, \boldsymbol{x}_{t-c+1} \;\; \cdots \;\; \boldsymbol{x}_{t-1} \;\; \boldsymbol{x}_t \,] \end{aligned}$$

- Assuming we have

  Σ' – the r largest singular values of M
  U' – the corresponding left singular vectors of M

  we would like to determine

  $\tilde{\Sigma}'$ – the r largest singular values of $\tilde{M}$
  $\tilde{U}'$ – the corresponding left singular vectors of $\tilde{M}$

  without performing a full SVD on $\tilde{M}$.

The IFAST Algorithm

1) $\boldsymbol{q}_1 = \frac{(I - U'U'^H)\,\boldsymbol{x}_{t-c}}{\|(I - U'U'^H)\,\boldsymbol{x}_{t-c}\|}$

2) $\boldsymbol{q}_2 = \frac{(I - [U' \;\; \boldsymbol{q}_1][U' \;\; \boldsymbol{q}_1]^H)\,\boldsymbol{x}_t}{\|(I - [U' \;\; \boldsymbol{q}_1][U' \;\; \boldsymbol{q}_1]^H)\,\boldsymbol{x}_t\|}$

3) $\tilde{F} = [U' \;\; Q]^H \tilde{M}\tilde{M}^H [U' \;\; Q]$

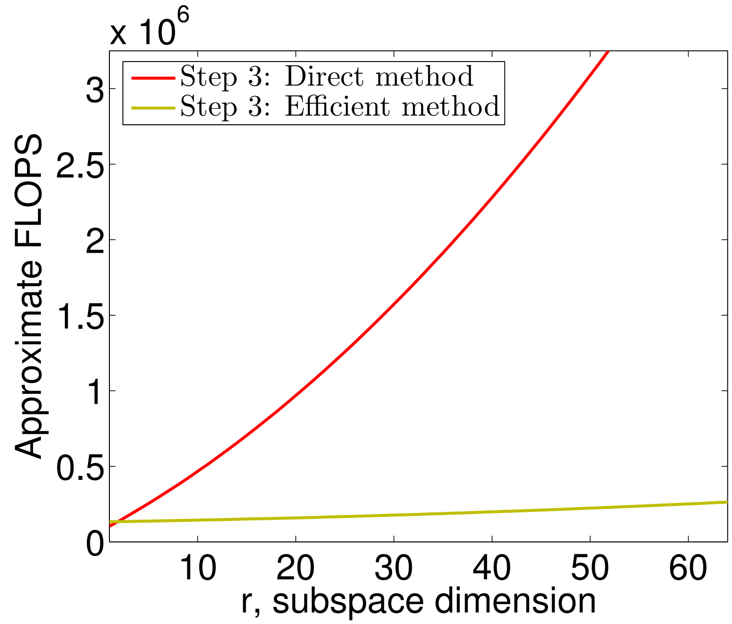
4) $U_F \Sigma_F U_F^H = \tilde{F}$

5) $\tilde{U}' = [U' \;\; Q] U_F$
$\tilde{\Sigma}' = \sqrt{\Sigma_F}$

The columns of the new matrix $\tilde{M}$, can be well approximated by projecting them onto the $r+2$ dimensional subspace which has an orthogonal basis consisting of the r columns of U' Gram-Schmidt augmented by $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.
After projecting the columns of $\tilde{M}$ onto this $r+2$ dimensional subspace, we perform a small $r+2$ dimensional SVD to obtain a good r-dimensional approximation subspace.

Efficiently Computing $\tilde{F}$

- Computing $\tilde{F}$ directly in step 3 is computationally costly because $\tilde{M}$ is an $n \times c$ matrix, and U' is an $n \times r$ matrix



- We can write $\tilde{F}$ as

$$\tilde{F} = \left[\begin{array}{c|c} \tilde{F}_c & U'^H \tilde{M} \tilde{M}^H Q \\ \hline Q^H \tilde{M} \tilde{M}^H U' & Q^H \tilde{M} \tilde{M}^H Q \end{array} \right]$$

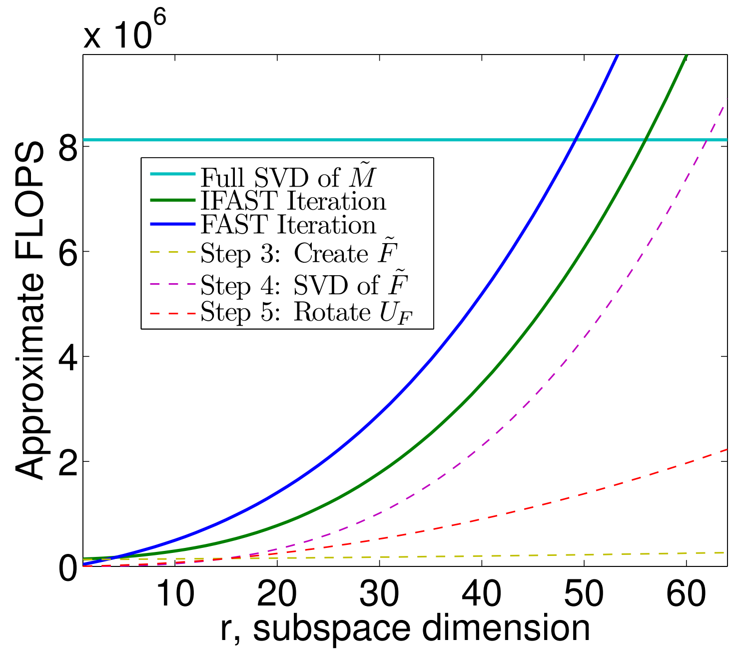
where the $r \times r$ matrix $\tilde{F}_c = U'^H \tilde{M} \tilde{M}^H U'$ is

$$\tilde{F}_c = \Sigma'^2 - U'^H \boldsymbol{x}_{t-c} \boldsymbol{x}_{t-c}^H U' + U'^H \boldsymbol{x}_t \boldsymbol{x}_t^H U'$$

which is the sum of a diagonal matrix plus two rank one matrices.

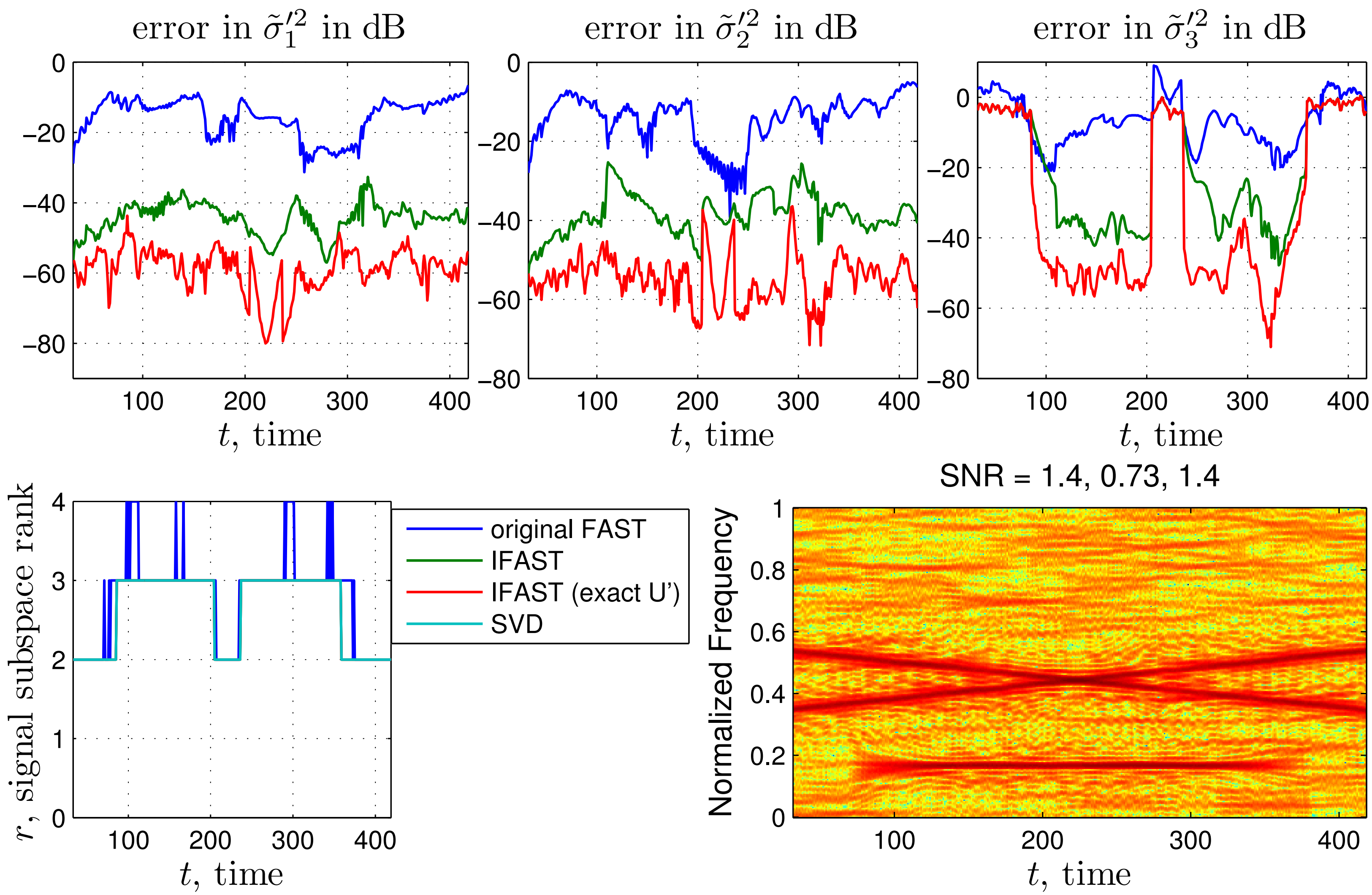
Computation

- $n = c = 64$, complex data
- Similar plot for other matrix dimensions
- Step 3 ($O(32nc)$ creation of $\tilde{F}$) dominates for $r < 15$
- Step 4 ($O(31r^3)$ SVD of $\tilde{F}$) dominates for $r > 15$
- Step 5 ($O(8nr^2)$ rotation of U_F) similar to step 4 for $r < 15$



- When the equivalent of $\tilde{F}$ for FAST is computed similarly to how it is done in IFAST, their computation is about the same.

Accuracy


error in $\tilde{\sigma}_1'^2$ in dB
error in $\tilde{\sigma}_2'^2$ in dB
error in $\tilde{\sigma}_3'^2$ in dB
t, time
t, time
t, time
r, signal subspace rank
t, time
original FAST
IFAST
IFAST (exact U')
SVD
SNR = 1.4, 0.73, 1.4
Normalized Frequency
t, time

IFAST Summary

Key Points:

- Accurate estimates of r largest singular values and corresponding left singular vectors
- Computational complexity of $O(nr^2)$
- No initial SVD required, can start with a single vector and grow M by making Q only one column

Additional Points:

- Robust to truncating r due to computational limitations
- Error in singular values proportional to error in angle for corresponding singular vector

Rectangular Window Update

- Given the SVD of $M = U\Sigma V^H$, and defining

$$\boldsymbol{a} = U^H \boldsymbol{x}_{t-c}, \qquad \boldsymbol{b} = U^H \boldsymbol{x}_t$$

  we can write

$$\tilde{G} = U^H \tilde{M} \tilde{M}^H U = \Sigma^2 - \boldsymbol{a}\boldsymbol{a}^H + \boldsymbol{b}\boldsymbol{b}^H$$

  which is a diagonal matrix plus two rank one matrices.

- The eigenvalues of $\tilde{G}$, which are the squares of the singular values of $\tilde{M}$, are the roots of the rank-two secular equation

$$w(\lambda) = \left(1 - \sum_{j=1}^{n} \frac{|a_j|^2}{\sigma_j^2 - \lambda}\right)\left(1 + \sum_{j=1}^{n} \frac{|b_j|^2}{\sigma_j^2 - \lambda}\right) + \left|\sum_{j=1}^{n} \frac{a_j^* b_j}{\sigma_j^2 - \lambda}\right|^2$$

Reduced Rank Approximation

- If we separate M into an r dimensional principal subspace, and the orthogonal $c - r$ dimensional one

$$M = [U' \;\; U^{\perp}] \begin{bmatrix} \Sigma' & 0 \\ 0 & \Sigma^{\perp} \end{bmatrix} [V' \;\; V^{\perp}]^H$$

- The eigenvalues values of $\tilde{G}' = U'^H \tilde{M} \tilde{M}^H U'$ are the roots of the rank-two secular equation

$$w'(\lambda) = \left(1 - \sum_{j=1}^{r} \frac{|a_j|^2}{\sigma_j^2 - \lambda}\right)\left(1 + \sum_{j=1}^{r} \frac{|b_j|^2}{\sigma_j^2 - \lambda}\right) + \left|\sum_{j=1}^{r} \frac{a_j^* b_j}{\sigma_j^2 - \lambda}\right|^2$$

which differs from the full secular equation only by the upper limit of the summation.

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IFAST Approximation

- Rotate the two columns of Q from steps one and two of the IFAST algorithm such that $\hat{\Sigma} = Q^H M M^H Q$ is diagonal

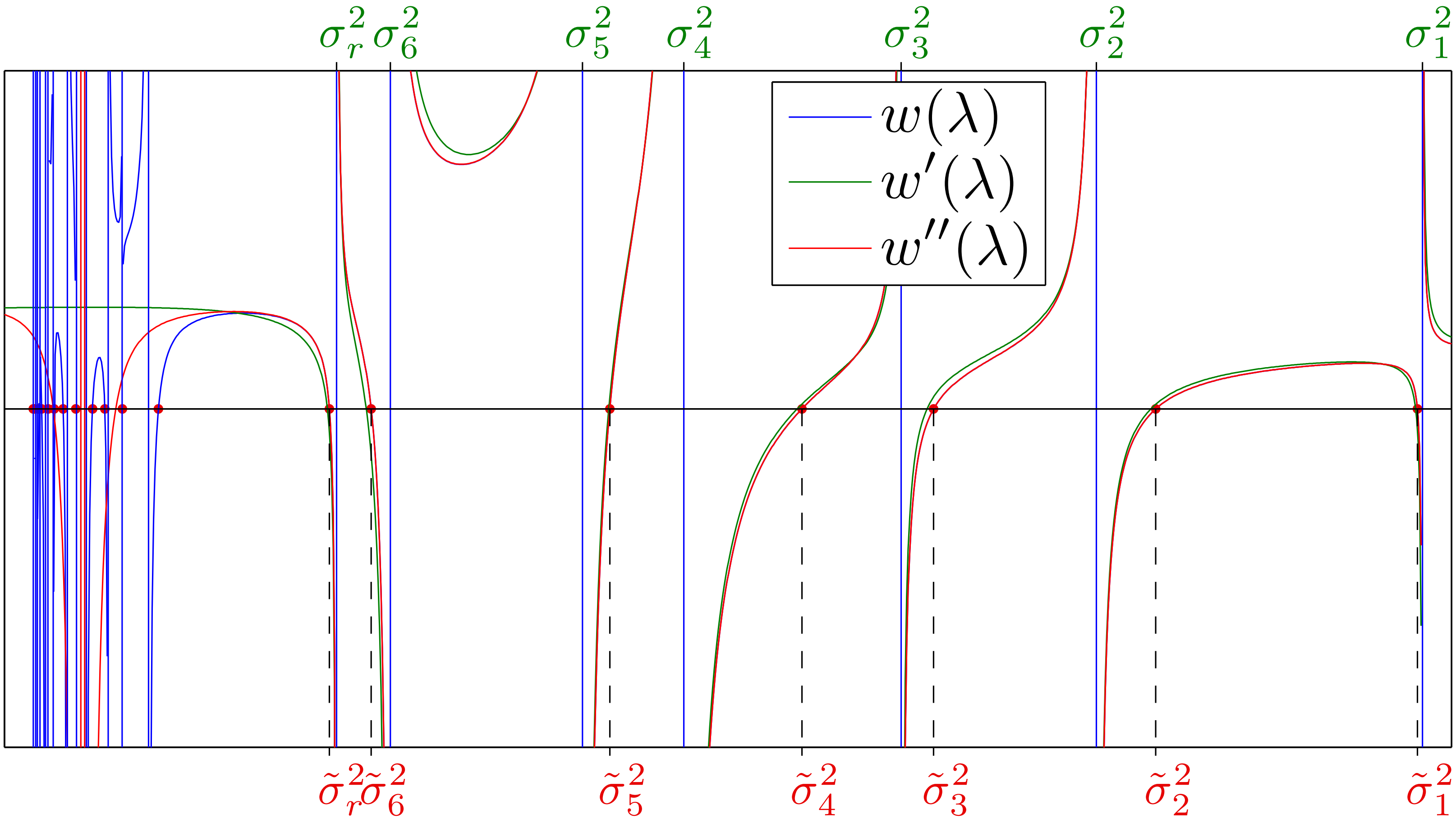
- Assume that U' is not an approximation, therefore $U'^H \tilde{M} \tilde{M}^H Q$ equals $\Sigma'^2 U'^H Q$, which must be zero

- The secular equation for $\tilde{F} = [U' \; Q]^H \tilde{M} \tilde{M}^H [U' \; Q]$, which we will call $w''(\lambda)$, is $w'(\lambda)$ with two additional terms in each summation

$$-\sum_{j=1}^{2} \frac{\left| \boldsymbol{q}_j^H \boldsymbol{x}_{t-c} \right|^2}{\hat{\sigma}_j - \lambda}, \qquad \sum_{j=1}^{2} \frac{\left| \boldsymbol{q}_j^H \boldsymbol{x}_t \right|^2}{\hat{\sigma}_j - \lambda}, \qquad \sum_{j=1}^{2} \frac{\boldsymbol{q}_j^H \boldsymbol{x}_t \boldsymbol{x}_{t-c}^H \boldsymbol{q}_j}{\hat{\sigma}_j - \lambda}$$

which are equal to the first two terms of binomial expansions of the $n - r$ missing terms of $w(\lambda)$

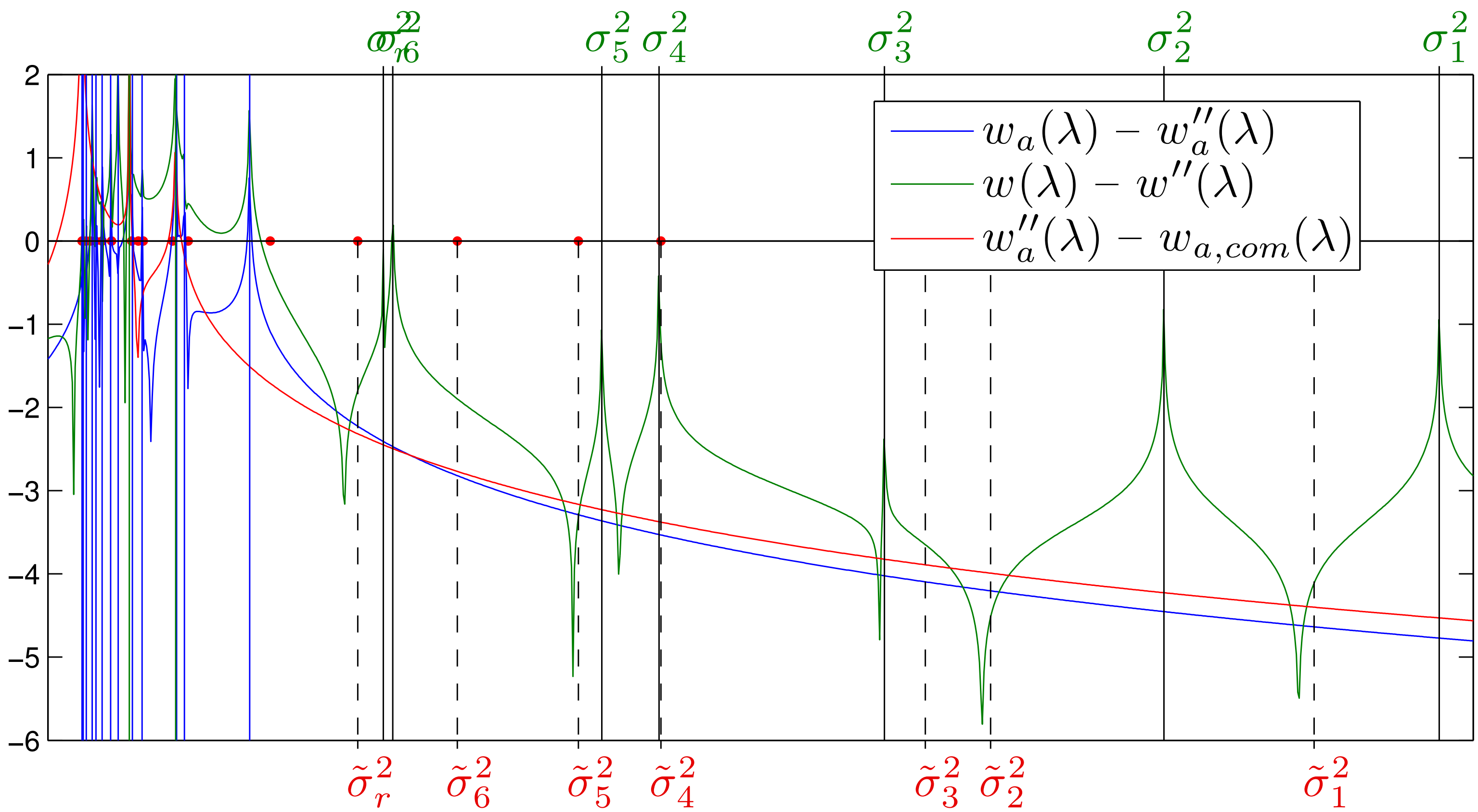
Rank Two Secular Function



The poles of $w(\lambda)$ are the squares of the singular values of M

The roots of $w(\lambda)$ are the squares of the singular values of $\tilde{M}$

Secular Function Differences



Error in secular function in $\log_{10}(|\text{difference}|)$

Analysis Summary

Key Points:

- Present the new rank-two secular equation, which is required to analyze the sliding window update eigenproblem
- Comparative analysis of full dimension secular equation with the secular equation for the IFAST algorithm
- Application of these results to show why IFAST has high accuracy

Additional Points:

- Method can be used to analyze any algorithm that can be written as a rank-two (or rank-one) modification to a diagonal matrix
- Can give estimate of error in each singular value estimate