WE DECIDE $\mathcal{H}_1$ IF

$$T(\underline{x}) = \underbrace{\sum_{n=0}^{N-1} x^2[n]}_{\text{ENERGY}} > \gamma'$$

OR $\frac{1}{N} T(\underline{x})$ = ESTIMATE OF VARIANCE

$\mathcal{H}_0$: $\text{VAR}(x[n]) = \sigma^2$
$\mathcal{H}_1$: $\text{VAR}(x[n]) = \sigma_s^2 + \sigma^2$

PERFORMANCE – ENERGY DETECTOR

ASIDE – SEE CHAPTER 2

PROBLEM: $\left.\begin{array}{l} X_1 \sim N(0,1) \\ X_2 \sim N(0,1) \end{array}\right\}$ INDEPENDENT

PDF OF $Y = X_1^2 + X_2^2$?

USE CDF APPROACH

$$\begin{aligned} F_Y(y) &= \Pr\{Y \le y\} \\ &= \Pr\{X_1^2 + X_2^2 \le y\} \end{aligned}$$

$$= \iint_{\{x_1^2+x_2^2 \le y\}} \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2+x_2^2)} \, dx_1 \, dx_2$$

LET $x_1 = r \cos\theta$

$x_2 = r \sin\theta$

$dx_1 \, dx_2 = r \, dr \, d\theta$

$$F_Y(y) = \int_0^{2\pi}\int_0^{\sqrt{y}} \frac{1}{2\pi} r e^{-\frac{1}{2}r^2} \, dr \, d\theta$$

$$= \int_0^{\sqrt{y}} r e^{-\frac{1}{2}r^2} \, dr$$

$$= -e^{-\frac{1}{2}r^2} \Big|_0^{\sqrt{y}}$$

$$= 1 - e^{-\frac{1}{2}y}$$

$$p(y) = dF_Y(y)/dy = \begin{cases} \frac{1}{2} e^{-\frac{1}{2}y} & y > 0 \\ 0 & y < 0 \end{cases}$$

(CENTRAL)

CALLED A CHI-SQUARED PDF

WITH 2 DEGREES OF FREEDOM

MORE GENERALLY IF

$$X = \sum_{i=1}^{n} x_i^2 \qquad x_i \sim N(0,1) \text{ IID}$$

$$p(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} \; x^{\nu/2-1} e^{-\frac{1}{2}x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt$$
$$= \text{GAMMA FUNCTION}$$

$$\Gamma(u) = (u-1)\,\Gamma(u-1)$$
$$\Rightarrow \quad \Gamma(n) = (n-1)! \quad n \text{ AN INTEGER}$$
$$\Gamma(1/2) = \sqrt{\pi}$$

$$E(x) = \nu$$
$$VAR(x) = 2\nu$$

WE WILL NEED RIGHT-TAIL PROBABILITY OF

$$Q_{\chi^2_\nu}(x) = \int_x^\infty p(t)\, dt \qquad x > 0$$

CAN BE SHOWN TO BE

$$Q_{\chi^2_\nu}(x) = e^{-\frac{1}{2}x} \sum_{k=0}^{\frac{\nu}{2}-1} \frac{(x/2)^k}{k!} \qquad \begin{matrix} \nu \text{ EVEN} \\ \nu \geq 2 \end{matrix}$$

$$Q_{\chi^2_\nu}(x) = \begin{cases} 2\,Q(\sqrt{x}) & \nu = 1 \\ 2\,Q(\sqrt{x}) + \dfrac{e^{-\frac{1}{2}x}}{\sqrt{\pi}} \displaystyle\sum_{k=1}^{\frac{\nu-1}{2}} \frac{(k-1)!\,(2x)^{k-1/2}}{(2k-1)!} & \begin{array}{l}\nu \text{ ODD} \\ \nu \geq 3\end{array} \end{cases}$$

USE Qchipr2.m IN APP. 2D
TO EVALUATE. (SPECIAL CASE
OF NONCENTRAL χ^2 PDF)

BACK TO ENERGY DETECTOR

$$T(\underline{x}) = \sum_{n=0}^{N-1} x^2[n]$$

$$x[n] \sim \begin{cases} \mathcal{N}(0, \sigma^2) & \mathcal{H}_0 \\ \mathcal{N}(0, \sigma_s^2 + \sigma^2) & \mathcal{H}_1 \end{cases}$$

$$\Rightarrow \quad \frac{x[n]}{\sqrt{\sigma^2}} \sim \mathcal{N}(0,1) \qquad \mathcal{H}_0$$

$$\frac{x[n]}{\sqrt{\sigma_s^2 + \sigma^2}} \sim \mathcal{N}(0,1) \qquad \mathcal{H}_1$$

AND ALL SAMPLES ARE IID

$$\frac{T(\underline{x})}{\sigma^2} \sim \chi_N^2 \qquad \mathcal{H}_0$$

$$\frac{T(\underline{x})}{\sigma_s^2+\sigma^2} \sim \chi_N^2 \qquad \mathcal{H}_1$$

$$P_{FA} = Pr\{T(\underline{x}) > \gamma' ; \mathcal{H}_0\}$$

$$= Pr\left\{\frac{T(\underline{x})}{\sigma^2} > \frac{\gamma'}{\sigma^2} ; \mathcal{H}_0\right\}$$

$$= Q_{\chi_N^2}(\gamma'/\sigma^2)$$

$$P_D = Q_{\chi_N^2}\left(\frac{\gamma'}{\sigma_s^2+\sigma^2}\right)$$

TO FIND THRESHOLD γ' OR

$$\gamma' = \sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})$$

SEE PROB. 5.1.

$$P_D = Q_{\chi_N^2}\left(\frac{\sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})}{\sigma_s^2+\sigma^2}\right)$$

$$= Q_{\chi_N^2}\left(Q^{-1}_{\chi_N^2}(P_{FA}) \frac{1}{\frac{\sigma_s^2}{\sigma^2}+1}\right)$$

PD INCREASES AS $\underbrace{\sigma_s^2/\sigma^2}_{SNR}$ INCREASES.

DIFFERENT SNR MEASURE THAN FOR KNOWN SIGNAL PROBLEM.

SEE FIG 5.1.

ALTERNATIVE VIEWPOINT

DECIDE $\mathcal{H}_1$ IF $T(\underline{x}) = \sum_n x^2[n] > \gamma'$

OR

$$T'(\underline{x}) = \sum_n x[n] \underbrace{\frac{\sigma_s^2}{\sigma_s^2+\sigma^2} x[n]}_{\hat{s}[n]} > \gamma''$$

$\hat{s}[n]$ IS AN ESTIMATE OF $s[n]$ UNDER $\mathcal{H}_1$.

FOR $\sigma_s^2/\sigma^2 \ll 1 \Rightarrow \hat{s}[n] \approx 0$

$\sigma_s^2/\sigma^2 \gg 1 \Rightarrow \hat{s}[n] \approx x[n]$

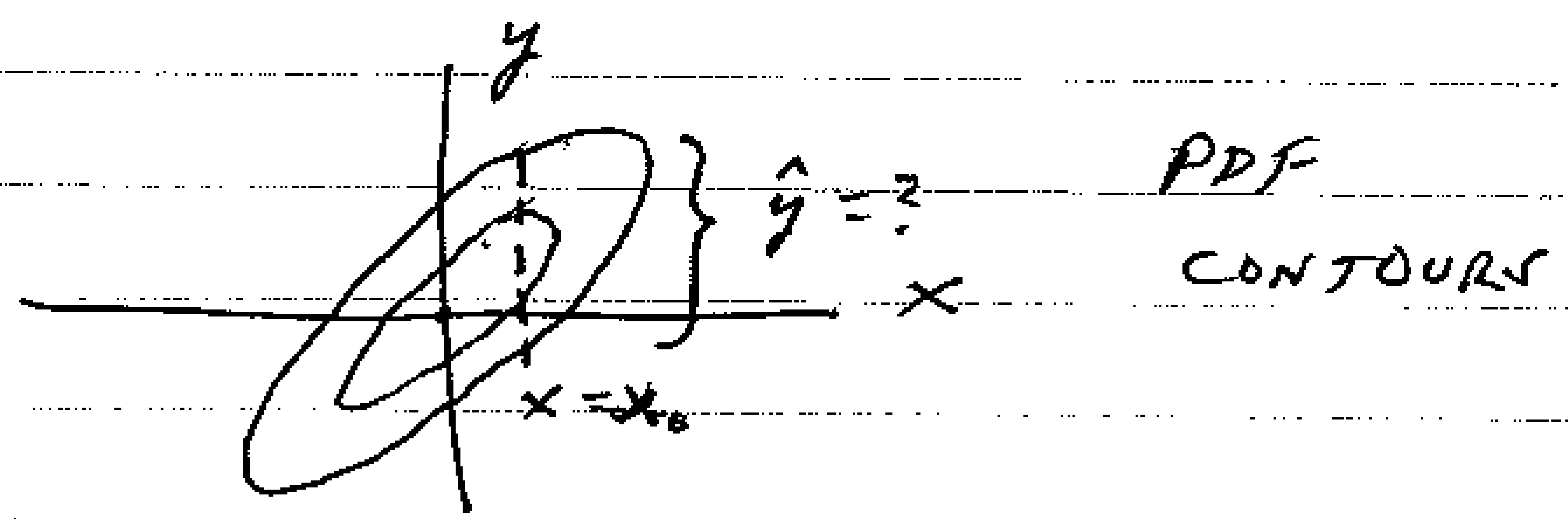
IN PARTICULAR $\hat{s}[n]$ IS MINIMUM MEAN SQUARE ERROR ESTIMATOR

$T'(\underline{x}) = \sum_n x(n)\,\hat{s}(n)$ CALLED

AN <u>ESTIMATOR-CORRELATOR</u>. TO SHOW THAT $\hat{s}(n)$ IS MMSE ESTIMATOR:

1) DEFINE MMSE
2) SHOW THAT MSE IS MINIMIZED BY CONDITIONAL MEAN
3) SHOW THAT CONDITIONAL MEAN EASILY EVALUATED FOR JOINTLY GAUSSIAN PDF

1) GIVEN x, y WHICH ARE RANDOM VARIABLES. PROBLEM IS TO ESTIMATE THE <u>REALIZATION</u> OF y IF WE KNOW THE <u>REALIZATION</u> OF x



$$MSE = E_{x,y}\left[(\hat{y} - y)^2\right]$$

$\uparrow$ FUNCTION OF x

$$= \iint (\hat{y}-y)^2 p(x,y)\, dx\, dy$$

2) TO MINIMIZE MSE

$$MSE = \iint (\hat{y}-y)^2 p(y|x)\, \overset{p(x)}{\wedge}\, dy\, dx$$

$$= \underbrace{\iint (\hat{y}(x)-y)^2 p(y|x)\, dy}\, p(x)\, dx$$

MINIMIZE FOR EACH X

$\Rightarrow$ MINIMIZES MSE

FOR A GIVEN X SAY $X = x_0$

$\hat{y}(x) = \hat{y}(x_0) = \text{CONSTANT} = c$

$$J = \int (c-y)^2 p(y|x_0)\, dy$$

$$\frac{\partial J}{\partial c} = \int 2(c-y) p(y|x_0)\, dy = 0$$

$$\Rightarrow \quad c \underbrace{\int p(y|x_0)\, dy}_{1} = \underbrace{\int y\, p(y|x_0)\, dy}_{E(y|x_0)}$$

$\hat{y}(x_0) = E(y|x_0)$ OR

$\therefore \quad \hat{y} = E(y|x)$ FOR ANY X

$$\hat{y} = \int y\, p(y/x)\, dy$$

= CONDITIONAL MEAN

= MEAN OF $p(y/x)$ PDF

3) NEED $E(y/x)$ FOR

$\begin{pmatrix} x \\ y \end{pmatrix}$ JOINTLY GAUSSIAN

ASSUME ZERO MEAN.

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\underline{0}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$

$$p(y/x) = p(x, y) / p(x)$$

$$= \frac{\dfrac{1}{2\pi \, \mathrm{DET}^{1/2}(\underline{C})} \, e^{-\frac{1}{2} \binom{x}{y}^T \underline{C}^{-1} \binom{x}{y}}}{\dfrac{1}{\sqrt{2\pi\sigma_x^2}} \, e^{-\frac{1}{2\sigma_x^2} x^2}}$$

$$= K e^{-\frac{1}{2} Q}$$

↑ CONSTANT

$$Q = \binom{x}{y}^T \underline{C}^{-1} \binom{x}{y} - x^2 / \sigma_x^2$$

BUT $\underline{C}^{-1} = \begin{bmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{bmatrix} \Big/ \left(\sigma_x^2 \sigma_y^2 - (\sigma_{xy})^2 \right)$

$$q = \frac{(x\ y)\begin{pmatrix}\sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}}{\sigma_x^2\sigma_y^2-(\sigma_{xy})^2} - \frac{x^2}{\sigma_x^2}$$

$$= \frac{\sigma_y^2 x^2 - 2\sigma_{xy}\,xy + \sigma_x^2 y^2}{\sigma_x^2\sigma_y^2-(\sigma_{xy})^2} - \frac{x^2}{\sigma_x^2}$$

$$= \underbrace{\frac{\sigma_x^2}{\sigma_x^2\sigma_y^2-(\sigma_{xy})^2}}_{1/\sigma_{y|x}^2} y^2 - \frac{2\sigma_{xy}}{\sigma_x^2\sigma_y^2-(\sigma_{xy})^2}xy + g(x)$$

$$= \frac{1}{\sigma_{y|x}^2}\left(y^2 - \frac{2\sigma_{xy}}{\sigma_x^2}xy\right) + g(x)$$

$$= \frac{1}{\sigma_{y|x}^2}\left(y^2 - 2\frac{\sigma_{xy}}{\sigma_x^2}xy + \frac{(\sigma_{xy})^2}{(\sigma_x^2)^2}x^2\right)$$

$$\underbrace{- \frac{1}{\sigma_{y|x}^2}\left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 x^2 + g(x)}_{h(x)}$$

$$= \frac{1}{\sigma_{y|x}^2}\left(y - \underbrace{\frac{\sigma_{xy}}{\sigma_x^2}x}_{\mu_{y|x}}\right)^2 + h(x)$$

$$p(y|x) = K e^{-\frac{1}{2}h(x)} \; e^{-\frac{1}{2\sigma_{y|x}^2}(y-\mu_{y|x})^2}$$

SINCE $\int p(y|x)\,dy = 1 \Rightarrow$

$$K e^{-\frac{1}{2}h(x)} = \frac{1}{\sqrt{2\pi\sigma_{y|x}^2}}$$

CONSTANT

AND THIS IS A GAUSSIAN PDF
WITH MEAN

$$E(y|x) = \mu_{y|x} = \frac{\sigma_{xy}}{\sigma_x^2}\, x$$

$\therefore$ MMSE ESTIMATOR OF y
BASED ON x FOR x, y
JOINTLY GAUSSIAN

ESTIMATION PROBLEM:

$$x(n) = s(n) + w(n)$$

"WGN", σ_s^2 ; WGN, σ^2

$s(n)$ INDEPENDENT OF $w(n)$

ESTIMATE $s(n)$ BASED ON $x(n)$

(BECAUSE OF INDEPENDENCE
$x[1], x[2], \ldots, x[n-1]$ CANNOT
HELP US ESTIMATE $s[0]$, ETC)

FOR ANY n THE MMSE ESTIMATOR
OF $s[n]$ IS

$$\hat{s}[n] = E(\underset{\uparrow y}{s[n]} \,/\, \underset{\uparrow x}{x[n]}).$$

SINCE $s[n], x[n]$ ARE JOINTLY
GAUSSIAN OR

$$\begin{bmatrix} s[n] \\ x[n] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} s[n] \\ w[n] \end{bmatrix}}$$

JOINTLY
GAUSSIAN
WHY?)

THUS,

$$\hat{s}[n] = \frac{\sigma_{xy}}{\sigma_x^2} x$$

$$= \frac{E(xy)}{E(x^2)} x$$

$$= \frac{E(x[n] \cdot s[n])}{E(x^2[n])} x[n]$$

$$= \frac{E(s^2[n])}{E(s^2[n]) + E(w^2[n])} x[n]$$

$$\therefore \quad \hat{s}(n) = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} x(n)$$

GENERAL ESTIMATOR - CORRELATOR

$$\underline{x} \sim N(\underline{0}, \sigma^2 \underline{I}) \qquad \mathcal{H}_0$$

$$N(\underline{0}, \underline{C}_s + \sigma^2 \underline{I}) \qquad \mathcal{H}_1$$

↑ BEFORE WE HAD

$$\underline{C}_s = \sigma_s^2 \underline{I}$$

TO FIND NP DETECTOR:

$$L(\underline{x}) = \frac{\frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(\underline{C}_s + \sigma^2 \underline{I})}}{\frac{1}{(2\pi\sigma^2)^{N/2}}}$$

$$\cdot \frac{e^{-\frac{1}{2}\underline{x}^T(\underline{C}_s + \sigma^2 \underline{I})^{-1}\underline{x}}}{e^{-\frac{1}{2\sigma^2}\underline{x}^T\underline{x}}} > \gamma$$

TAKE LN'S AND KEEP $\underline{x}$ TERMS

$$-\frac{1}{2} \underline{x}^T \left((\underline{C}_s + \sigma^2 \underline{I})^{-1} - \frac{1}{\sigma^2} \underline{I} \right) \underline{x} > \gamma'$$

OR

$$T(\underline{x}) = \sigma^2 \underline{x}^T \left[\frac{1}{\sigma^2} \underline{I} - (\underline{C}_s + \sigma^2 \underline{I})^{-1} \right] \underline{x}$$

USE THE MATRIX INVERSION LEMMA:

$$(\underbrace{\underline{A}}_{\sigma^2 \underline{I}} + \underbrace{\underline{B}}_{\underline{I}}\underbrace{\underline{C}}_{\underline{C}_s}\underbrace{\underline{D}}_{\underline{I}})^{-1} = \underline{A}^{-1} - \underline{A}^{-1}\underline{B}(\underline{D}\underline{A}^{-1}\underline{B} + \underline{C}^{-1})^{-1}\underline{D}\underline{A}^{-1}$$

$$= \frac{1}{\sigma^2}\underline{I} - \frac{1}{\sigma^4}\underline{I}\left(\frac{1}{\sigma^2}\underline{I} + \underline{C}_s^{-1}\right)^{-1}$$

$$T(\underline{x}) = \underline{x}^T\left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\underline{I} + \underline{C}_s^{-1}\right)^{-1}\right]\underline{x}$$

$$= \underline{x}^T \frac{1}{\sigma^2}\left[\left(\frac{1}{\sigma^2}(\underline{C}_s + \sigma^2\underline{I})\underline{C}_s^{-1}\right)^{-1}\right]\underline{x}$$

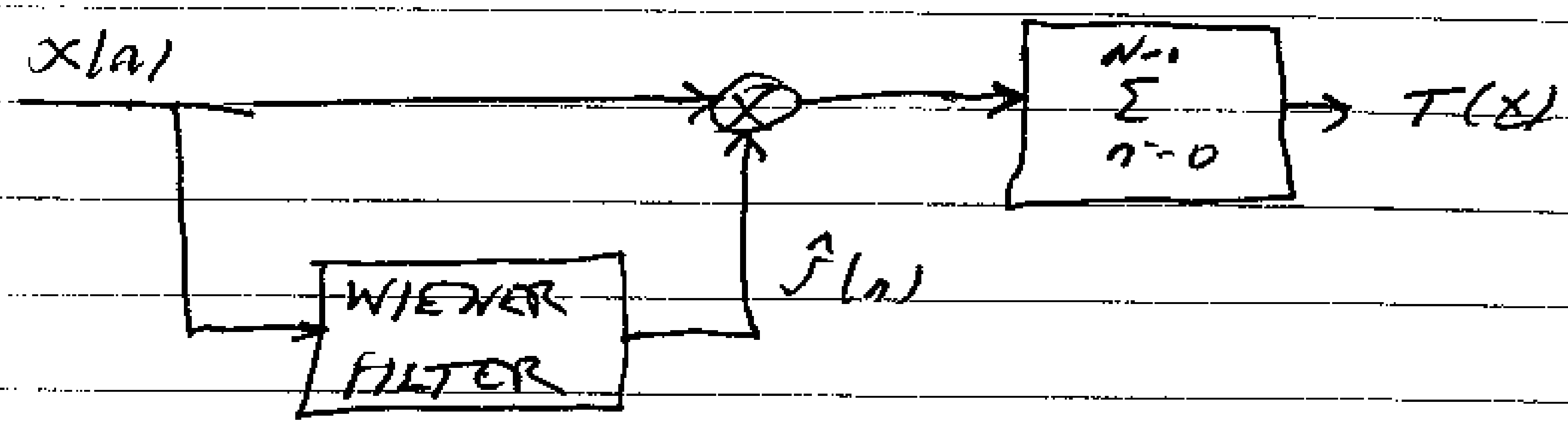
$$= \underline{x}^T \underbrace{\underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}}_{\hat{\underline{s}}}$$

$\hat{\underline{s}}$ = MMSE ESTIMATOR OF $\underline{s}$

ALSO, CALLED A WIENER FILTER.

$$T(\underline{x}) = \sum_{n=0}^{N-1} x[n]\,\hat{s}[n]$$

= ESTIMATOR - CORRELATOR



$$\hat{\underline{s}} = \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}$$

PERFORMANCE IS DIFFICULT TO OBTAIN ANALYTICALLY

$$T(\underline{x}) = \underline{x}^T\underline{A}\,\underline{x} \qquad \underline{A} = \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}$$

QUADRATIC FORM – NO LONGER LINEAR IN $\underline{x}$ (AS IN MATCHED FILTER)

A CANONICAL FORM IS FOUND AS FOLLOWS:

LET $\underline{V}^T\underline{C}_s\underline{V} = \underline{\Lambda}_s$ EIGENDECOMPOSITION

MODAL MATRIX

$\mathrm{DIAG}(\lambda_{s_0}, \lambda_{s_1}, \ldots, \lambda_{s_{N-1}})$

$$\underline{V}^T\underline{V} = \underline{I} \qquad \underline{V} = [\underline{v}_0\ \underline{v}_1\ \ldots\ \underline{v}_{N-1}]$$

$$\underset{\uparrow}{\underline{C_S}\, \underline{V_i}} = \underset{\uparrow}{\lambda_i}\, \underline{V_i}$$

EIGENVECTOR EIGENVALUE

THIS CAN BE USED TO DIAGONALIZE $\underline{A}$.

$$T(\underline{X}) = \underline{X}^T \underline{C_S}(\underline{C_S} + \sigma^2 \underline{I})^{-1} \underline{X}$$

$$= \underline{X}^T \underline{V}\, \underline{\Lambda_S}\, \underline{V}^T (\underline{V}\, \underline{\Lambda_S}\, \underline{V}^T + \sigma^2 \underline{I})^{-1} \underline{X}$$

SINCE $\underline{V}^T \underline{C_S} \underline{V} = \underline{\Lambda_S}$

$$\Rightarrow \underline{V}^T \underline{C_S} = \underline{\Lambda_S} \underline{V}^{-1}$$

$$\underline{C_S} = \underline{V}^{T^{-1}} \underline{\Lambda_S} \underline{V}^{-1}$$

$$= \underline{V}^{-1^T} \underline{\Lambda_S} \underline{V}^{-1}$$

$$= \underline{V}\, \underline{\Lambda_S}\, \underline{V}^T$$

SINCE $\underline{V}^{-1} = \underline{V}^T$ ORTHOGONAL MATRIX

NOW LET $\underline{y} = \underline{V}^T \underline{X}$

$$T(\underline{x}) = \underline{y}^T \underline{\Lambda}_s \left(\left(\underline{V}\underline{\Lambda}_s\underline{V}^T + \sigma^2 \underline{I} \right)\underline{V} \right)^{-1} \underbrace{\underline{V}\underline{V}^T\underline{x}}_{\underline{y}}$$

$$= \underline{y}^T \underline{\Lambda}_s \left(\underline{V}\underline{\Lambda}_s + \sigma^2 \underline{V} \right)^{-1} \underline{V}\, \underline{y}$$

$$= \underline{y}^T \underline{\Lambda}_s \left[\underline{V}^{-1} \left(\underline{V}\underline{\Lambda}_s + \sigma^2 \underline{V} \right) \right]^{-1} \underline{y}$$

$$= \underline{y}^T \underline{\Lambda}_s \left(\underline{\Lambda}_s + \sigma^2 \underline{I} \right)^{-1} \underline{y}$$

$$= \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2}\, y^2[n]$$

CANONICAL FORM (WEIGHTED ENERGY DETECTOR)

NOTE THAT $\underline{y} = \underline{V}^T \underline{x}$

$$\sim \mathcal{N}\left(\underline{0}, \underline{V}^T \underline{C}_x \underline{V}\right)$$

BUT UNDER $\mathcal{H}_0$ $\quad \underline{C}_{\underline{x}} = \sigma^2 \underline{I}$

$$\Rightarrow \quad \underline{y} \sim \mathcal{N}\left(\underline{0}, \sigma^2 \underline{I}\right) \quad \mathcal{H}_0$$

UNDER $\mathcal{H}_1$ $\quad \underline{C}_{\underline{x}} = \underline{C}_s + \sigma^2 \underline{I}$

$$\underline{V}^T \left(\underline{C}_s + \sigma^2 \underline{I} \right) \underline{V} = \underline{\Lambda}_s + \sigma^2 \underline{I}$$

$$\Rightarrow \underline{y} \sim N(\underline{0}, \underline{\Lambda}_s + \sigma^2 \underline{I}) \quad \mathcal{H}_1$$

THUS, $y[n] \sim N(0, \sigma^2) \quad \mathcal{H}_0$

$N(0, \lambda_{s_n} + \sigma^2) \quad \mathcal{H}_1$

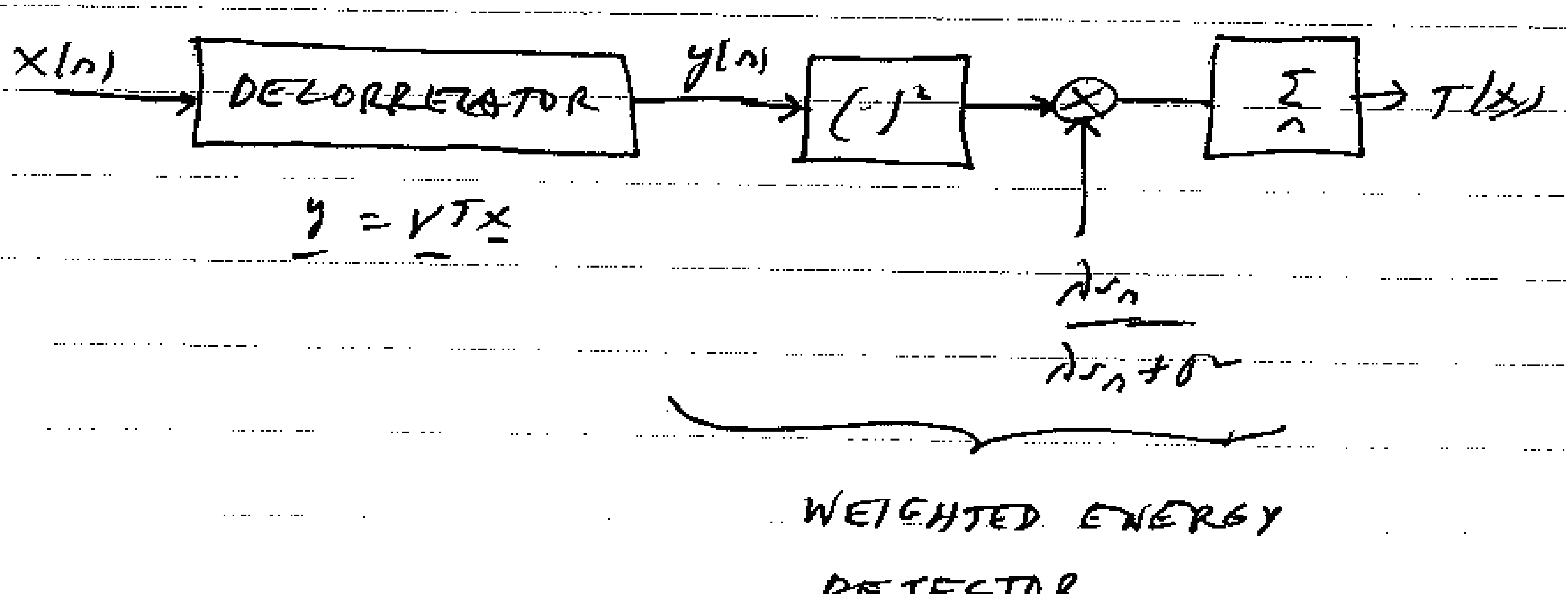
AND ALL UNCORRELATED AND HENCE INDEPENDENT.

$$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$$

= WEIGHTED SUM OF IND. χ_1^2 RANDOM VARIABLES

SEE 5.10, 5.11 FOR P_{FA}, P_D.

NOTE THAT $\underline{y} = \underline{V}^T \underline{x}$ DECORRELATES RANDOM VARIABLES.



LINEAR MODEL (BAYESIAN)

EXAMPLE : RANDOM DC LEVEL IN WGN

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = A + w[n]$$

NOW ASSUME A IS RANDOM WITH $A \sim \mathcal{N}(0, \sigma_A^2)$ AND IND. OF $w[n]$

THEN
$$\underline{x} = \underline{w} \qquad \mathcal{H}_0$$
$$\underline{x} = \underbrace{\underline{1}\, A}_{\underline{s} = \underline{H}\underline{\theta}} + \underline{w} \qquad \mathcal{H}_1$$

CALLED THE BAYESIAN LINEAR MODEL. UNDER $\mathcal{H}_1$

$$\underline{x} \sim \mathcal{N}(\underline{0}, \underline{C}_s + \sigma^2 \underline{I})$$

WHERE
$$\underline{C}_s = E(\underline{s}\,\underline{s}^T)$$
$$= E(\underline{1} A (\underline{1} A)^T)$$
$$= E(A^2)\, \underline{1}\,\underline{1}^T = \sigma_A^2\, \underline{1}\,\underline{1}^T$$

MORE GENERALLY

$$\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$$

$\underline{x}$: $N \times 1$, $\underline{H}$: $N \times p$, $\underline{\theta}$: $p \times 1$, $\underline{w}$: $N \times 1$

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

$\underline{H}$ IS KNOWN

$\underline{\theta}$ IS RANDOM AND

$\underline{\theta} \sim N(\underline{0}, \underline{C}_\theta)$

$\underline{\theta}$ IS IND OF $\underline{w}$

TO FIND THE NP DETECTOR:

$$\mathcal{H}_0 : \underline{x} = \underline{w}$$

$$\mathcal{H}_1 : \underline{x} = \underbrace{\underline{H}\,\underline{\theta}}_{\underline{s}} + \underline{w}$$

$\underline{s}$ ← RANDOM SIGNAL

THE ESTIMATOR - CORRELATOR IS

$$T(\underline{x}) = \underline{x}^T \underline{C}_s (\underline{C}_s + \sigma^2 \underline{I})^{-1} \underline{x}$$

BUT $\underline{C}_s = E(\underline{s}\,\underline{s}^T)$

$$= E(\underline{H}\,\underline{\theta}\,\underline{\theta}^T \underline{H}^T)$$

$$= \underline{H}\,\underline{C}_\theta \underline{H}^T$$

$$T(\underline{x}) = \underline{x}^T \underline{H}\,\underline{C}_\theta \underline{H}^T (\underline{H}\,\underline{C}_\theta \underline{H}^T + \sigma^2 \underline{I})^{-1} \underline{x}$$