BUT $p(\underline{x}) = \sum_j p(\underline{x}|H_j)P(H_j)$

$= p(\underline{x}|H_k)P(H_k)$

$+ \sum_{\substack{j \\ j\neq k}} p(\underline{x}|H_j)P(H_j)$

$$\underbrace{\sum_{\substack{j \\ j\neq k}} p(\underline{x}|H_j)P(H_j)}_{\text{MINIMIZED}} = p(\underline{x}) - \underbrace{p(\underline{x}|H_k)P(H_k)}_{\text{MAXIMIZED}}$$

DECIDE H_k IF $p(\underline{x}|H_i)P(H_i)$ IS MAXIMIZED FOR $i=k$ OR

$$P(H_k|\underline{x}) > P(H_i|\underline{x}) \quad i\neq k$$

WHAT IS RESULT FOR $M=2$?

<u>SPECIAL CASE</u>: $P(H_i) = 1/M$

$$P(H_i|\underline{x}) = \frac{p(\underline{x}|H_i)\overbrace{P(H_i)}^{1/M}}{p(\underline{x})}$$

$\Rightarrow$ MAXIMIZE $p(\underline{x}|H_i)$ ML RULE

EQUIVALENTLY MAP RULE DECIDES $\mathcal{H}_k$ IF

$$\ln p(\underline{x} | \mathcal{H}_k) + \ln P(\mathcal{H}_k)$$

IS MAXIMUM.

SEE EXAMPLE 3.6.

CHAPTER 4 – DETERMINISTIC SIGNALS

$$\mathcal{H}_0 : \quad x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : \quad x[n] = \underbrace{s[n]}_{\text{KNOWN DETERMINISTIC SIGNAL}} + \underbrace{w[n]}_{\text{WGN}} \qquad n = 0, 1, \ldots, N-1$$

TO MAXIMIZE P_D FOR GIVEN P_{FA} DECIDE $\mathcal{H}_1$ IF

$$L(\underline{x}) = \frac{p(\underline{x}; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$$

$$p(\underline{x};\mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n])^2}$$

$$p(\underline{x};\mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]}$$

$$L(\underline{x}) = e^{-\frac{1}{2\sigma^2}\left[-2\sum_n x[n]s[n] + \sum_n s^2[n]\right]}$$

TAKE LN

$$\frac{1}{\sigma^2}\sum_n x[n]s[n] - \frac{1}{2\sigma^2}\sum_n s^2[n] > \ln\gamma$$

NOT A FUNCTION OF DATA

$$\frac{1}{\sigma^2}\sum_n x[n]s[n] > \ln\gamma + \frac{1}{2\sigma^2}\sum_n s^2[n]$$

OR

$$T(\underline{x}) = \sum_n x[n]s[n] > \underbrace{\sigma^2\ln\gamma + \frac{1}{2}\sum s^2[n]}_{\gamma'}$$

<u>EXAMPLE</u> : DC LEVEL IN WGN

$s[n] = A \quad , \quad A > 0$

$$T(\underline{x}) = A\sum_n x[n] > \gamma'$$

OR

$$T'(\underline{x}) = \frac{1}{NA}T(\underline{x}) = \frac{1}{N}\sum_n x[n] > \gamma''$$

SAMPLE MEAN

WHAT HAPPENS IF $A < 0$?

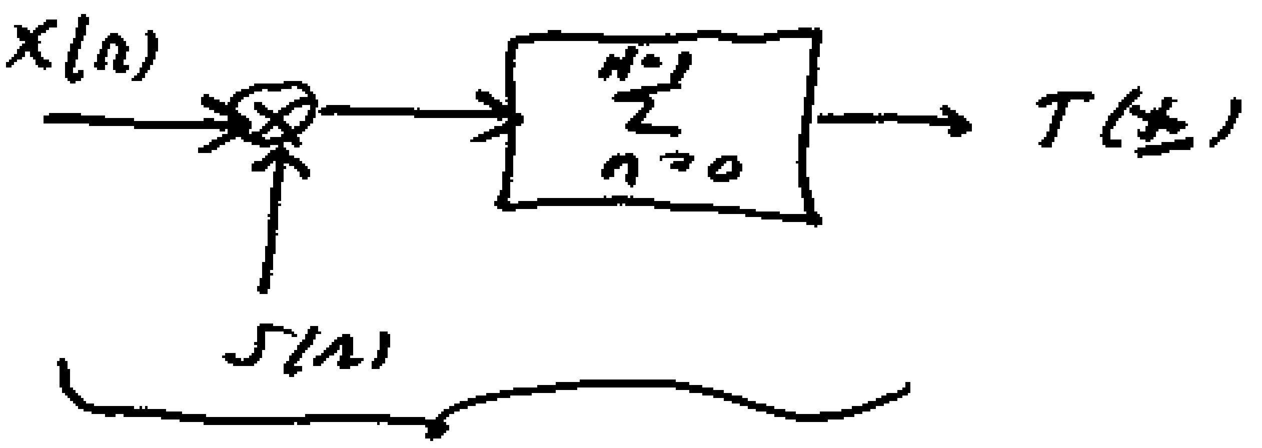
NOTE THAT IN GENERAL WE WEIGHT SAMPLES BY $s[n]$

$$T(\underline{x}) = \sum_n x[n]\, s[n]$$

↑ WEIGHTS

WHY?

$$SNR = \frac{s^2[n]}{E(w^2[n])} = \frac{s^2[n]}{\sigma^2}$$



REPLICA-CORRELATOR

SINCE $T(\underline{x})$ IS LINEAR FUNCTION OF $\underline{x}$ ⇒ CAN VIEW AS LINEAR FILTER

$$T(\underline{x}) = \sum_{k=0}^{N-1} s[k]\, x[k]$$

$$y[n] = \sum_{k=0}^{n} h[n-k]\, x[k]$$

AT $n = N-1$

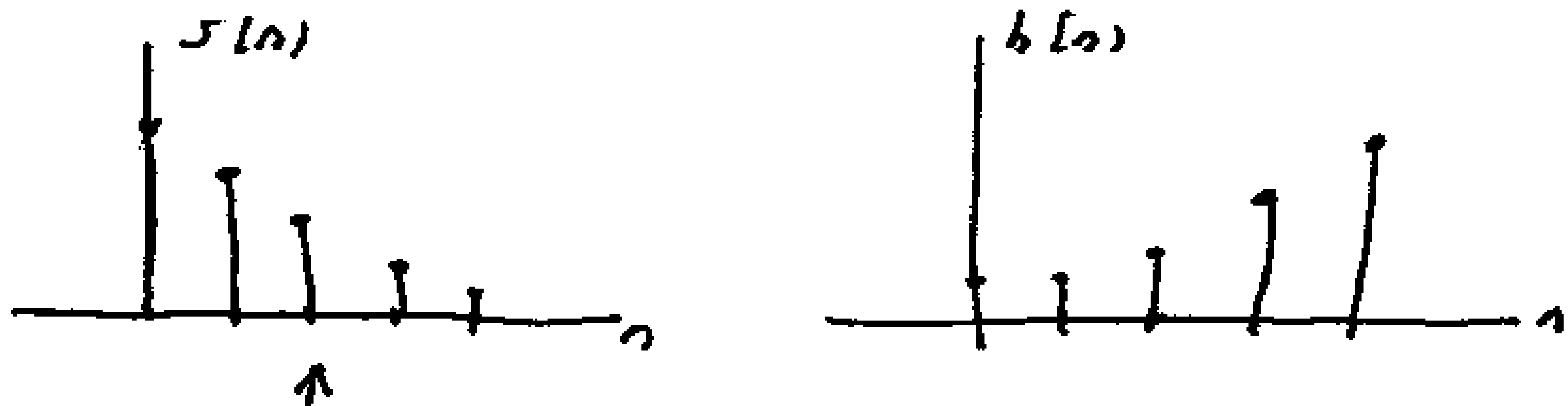
$$y[N-1] = \sum_{k=0}^{N-1} \underbrace{h[N-1-k]}_{s[k]}\, x[k]$$

LET $s[k] = h[N-1-k] \qquad k = 0, 1, \ldots, N-1$

OR $h[k] = s[N-1-k]$

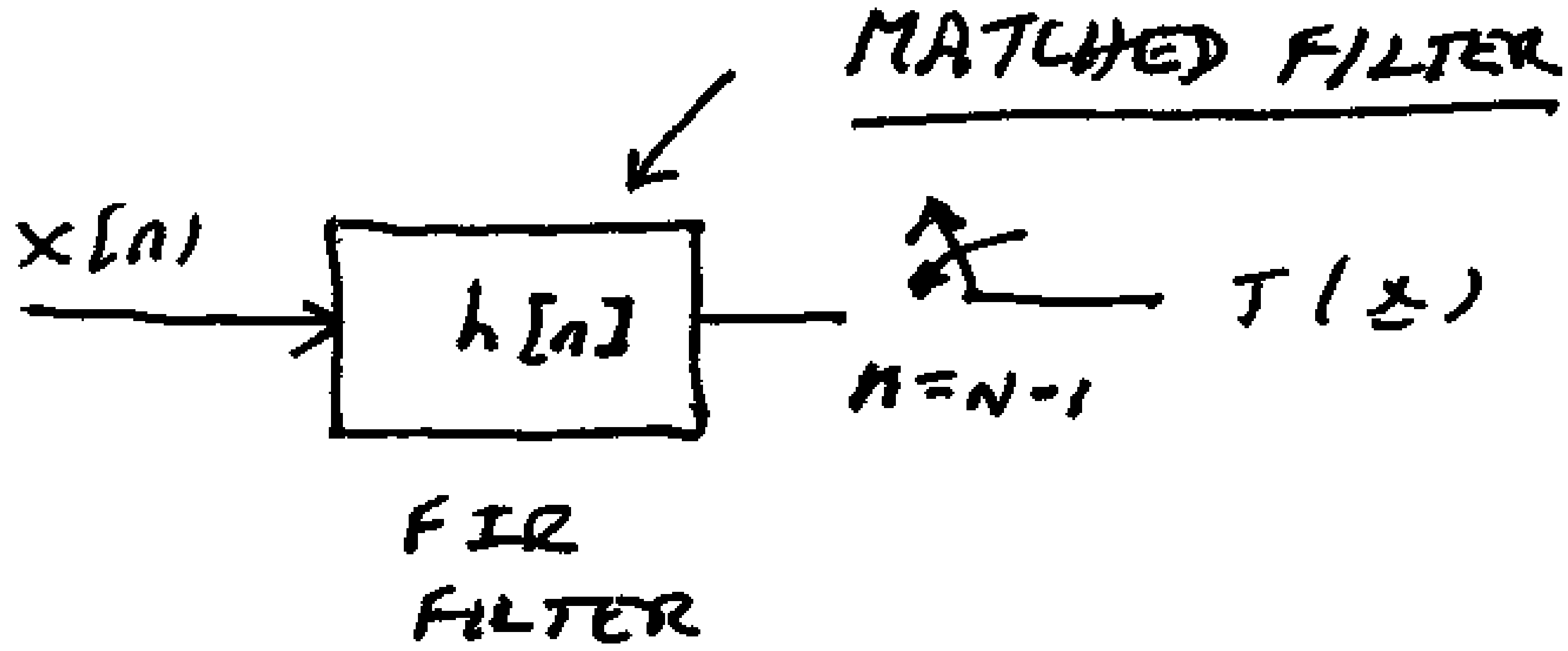
OR $h[n] = s[N-1-n] \qquad n = 0, 1, \ldots, N-1$

<u>EXAMPLE</u> : $N = 5$



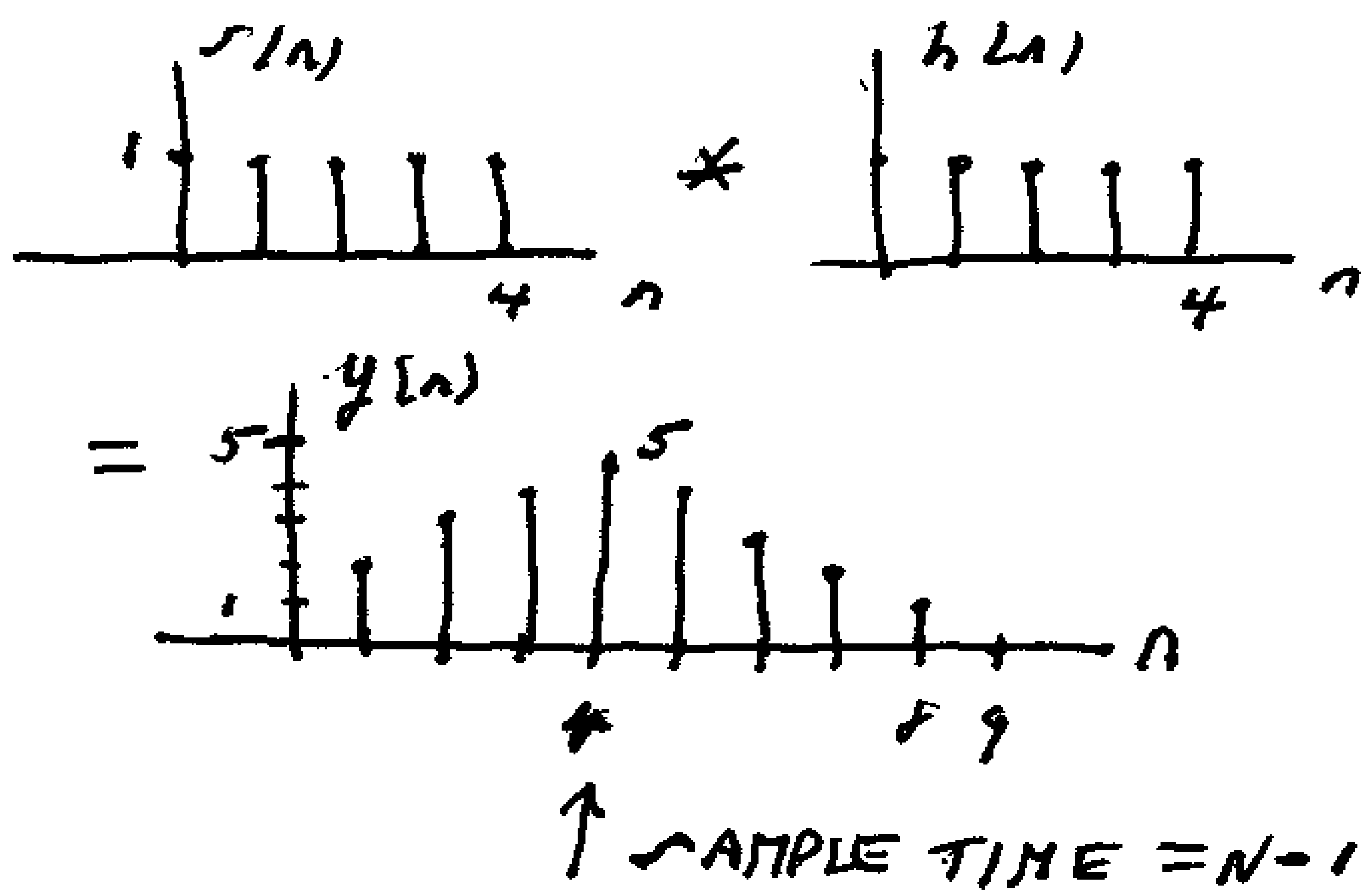
FLIP ABOUT
$n=0$ AND SHIFT
RIGHT $(N-1)$ SAMPLES

$h[n]$ IS <u>MATCHED</u> TO $s[n]$



<u>EXAMPLE</u> : DC LEVEL IN WGN

$s[n] = 1, \; N = 5$

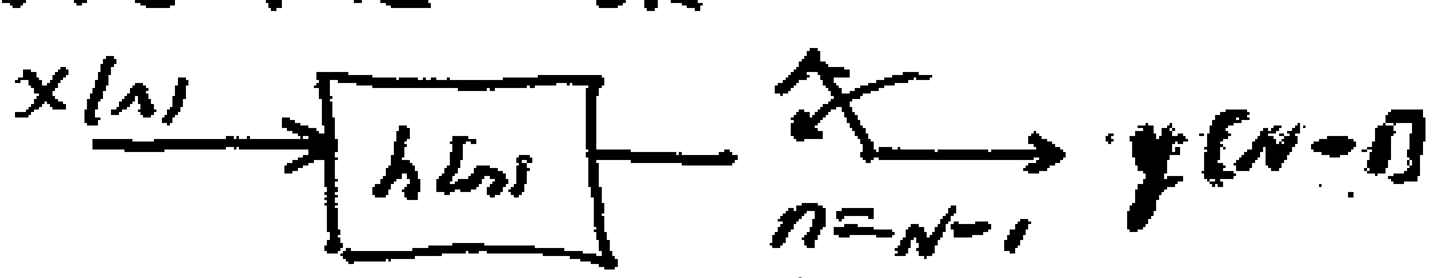


IMPORTANT TO SAMPLE AT CORRECT TIME

WILL CONSIDER UNKNOWN ARRIVAL TIME LATER!

<u>MAXIMUM SNR PROPERTY</u>

CONSIDER ALL POSSIBLE FIR FILTER DETECTORS OR



h(n) NONZERO OVER [0, N−1].
WHICH h(n) MAXIMIZES SNR AT ITS OUTPUT?

$$SNR = \frac{E^2(y(N-1))}{VAR(y(N-1))}$$

DO NOT ASSUME $h(n) = s(N-1-n)$.
TO MAXIMIZE SNR LET $\underline{h} = [h(N-1) \ldots h(0)]^T$
$\underline{s} = [s(0) \ldots s(N-1)]^T$.

$$SNR = \frac{\left(\sum_{k=0}^{N-1} h(N-1-k)\, s(k)\right)^2}{E\left[\left(\sum_{k=0}^{N-1} h(N-1-k)\, w(k)\right)^2\right]} = \frac{\text{SIGNAL POWER}}{\text{NOISE POWER}}$$

AT FILTER OUTPUT

$$SNR = \frac{(\underline{h}^T\underline{s})^2}{E[(\underline{h}^T\underline{w})^2]} = \frac{(\underline{h}^T\underline{s})^2}{E[\underline{h}^T\underline{w}\,\underline{w}^T\underline{h}]}$$

$$= \frac{(\underline{h}^T\underline{s})^2}{\underline{h}^T E(\underline{w}\,\underline{w}^T)\underline{h}} = \frac{(\underline{h}^T\underline{s})^2}{\sigma^2\, \underline{h}^T\underline{h}}$$

BUT BY CAUCHY-SCHWARZ INEQUALITY

$$(\underline{h}^T \underline{s})^2 \leq (\underline{h}^T \underline{h})(\underline{s}^T \underline{s})$$

WITH $=$ IF AND ONLY IF $\underline{h} = c\underline{s}$ FOR c A CONSTANT.

SIGNAL ENERGY

$$\Rightarrow \quad SNR \leq \frac{1}{\sigma^2} \underline{s}^T \underline{s} = \mathcal{E}/\sigma^2$$

FOR $=$ OR MAXIMUM SNR

$$\underline{h} = c\underline{s} \quad \text{OR}$$

$$h[n] = c\, s[N-1-n] \qquad n = 0, 1, \ldots, N-1$$

THUS, MATCHED FILTER MAXIMIZES SNR AT OUTPUT OF ALL FIR FILTERS.

WHAT IF NOISE IS NONGAUSSIAN?

PERFORMANCE OF MATCHED FILTER

DECIDE $\mathcal{H}_1$ IF $T(\underline{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$

WISH TO DETERMINE γ' AND PERFORMANCE

$T(\underline{x})$ IS GAUSSIAN SINCE $x[n]$ IS GAUSSIAN

$$E(T;\mathcal{H}_0) = E\left(\sum_n w(n)s(n)\right) = 0$$

$$E(T;\mathcal{H}_1) = E\left(\sum_n (s(n)+w(n))s(n)\right)$$

$$= \mathcal{E}$$

$$\mathrm{VAR}(T;\mathcal{H}_0) = \mathrm{VAR}(T;\mathcal{H}_1)$$

$$= \mathrm{VAR}\left(\sum_n w(n)s(n)\right)$$

$$= \sum_n s^2(n)\,\mathrm{VAR}(w(n))$$

$$= \sigma^2 \sum s^2(n) = \sigma^2\mathcal{E}$$

$$\Rightarrow \qquad T \sim \begin{array}{ll} \mathcal{N}(0, \sigma^2\mathcal{E}) & \mathcal{H}_0 \\ \mathcal{N}(\mathcal{E}, \sigma^2\mathcal{E}) & \mathcal{H}_1 \end{array}$$

$$P_{FA} = P_r\{T > \gamma';\mathcal{H}_0\}$$

$$= Q\left(\gamma'/\sqrt{\sigma^2\mathcal{E}}\right)$$

$$P_D = P_r\{T > \gamma';\mathcal{H}_1\}$$

$$= Q\left(\frac{\gamma'-\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right)$$

TO SET P_{FA} CHOOSE

$$\gamma' = \sqrt{\sigma^2 \mathcal{E}}\, Q^{-1}(P_{FA})$$

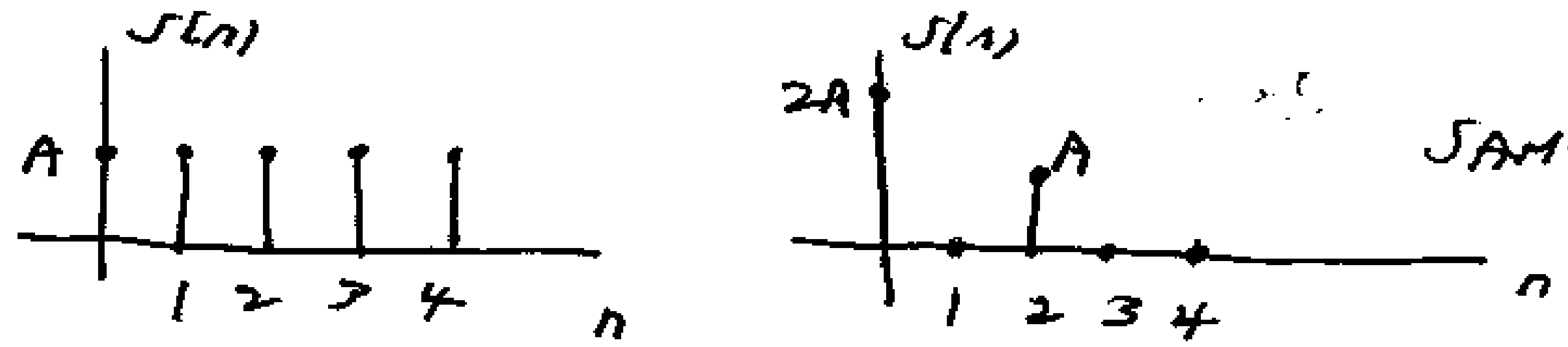
TO FIND PERFORMANCE

$$P_D = Q\left(\frac{\gamma'}{\sqrt{\sigma^2 \mathcal{E}}} - \sqrt{\mathcal{E}/\sigma^2}\right)$$

$$= Q\Big(Q^{-1}(P_{FA}) - \sqrt{\underbrace{\mathcal{E}/\sigma^2}_{\eta}}\Big)$$

η = ENERGY-TO-NOISE RATIO

PERFORMANCE ONLY DEPENDS ON SIGNAL ENERGY NOT SHAPE



SAME P_D

ALSO NOTE THAT $\mathcal{E}/\sigma^2$ IS SNR AT MATCHED FILTER OUTPUT.

SHOULD SHOW THAT $\eta = d^2$ (DEFLECTION COEFF.)

GENERALIZED MATCHED FILTERS

NOW ASSUME NOISE IS GAUSSIAN BUT NOT WHITE.

$\Rightarrow \quad \underline{C} \neq \sigma^2 \underline{I}$ $\qquad \underline{C}$ = COVARIANCE MATRIX OF $w(n)$

EXAMPLE : $N = 3$

$$\underline{C} = \begin{bmatrix} E(w^2[0]) & E(w[0]w[1]) & E(w[0]w[2]) \\ E(w[1]w[0]) & E(w^2[1]) & E(w[1]w[2]) \\ E(w[2]w[0]) & E(w[2]w[1]) & E(w^2[2]) \end{bmatrix}$$

FOR WHITE NOISE

$$E(w[m]w[n]) = \begin{cases} \sigma^2 & m = n \\ 0 & m \neq n \end{cases}$$

$\Rightarrow \quad \underline{C} = \sigma^2 \underline{I}$

EXAMPLE : $N = 3$ COLORED NOISE (WSS)

$$E(w[m]w[n]) = r_w(m-n) = a^{|m-n|}$$

$\uparrow$ AUTOCORRELATION

$$\underline{C} = \begin{bmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{bmatrix} \qquad |a| < 1$$

NOW FOR ANY $\underline{C}$ (MUST BE SYMMETRIC AND POSITIVE DEFINITE)

$$\underline{C} = E\left[(\underline{w} - \underbrace{E(\underline{w})}_{=0})(\underline{w} - \underbrace{E(\underline{w})}_{=0})^T\right]$$

$$= E(\underline{w}\,\underline{w}^T)$$

DETERMINE N-P DETECTOR FOR

$$\mathcal{H}_0 : x(n) = w(n)$$

$$\mathcal{H}_1 : x(n) = \underbrace{s(n)}_{\text{KNOWN}} + w(n)$$

$$\underline{w} = [w(0)\, w(1) \ldots w(N-1)]^T \qquad N \times 1$$

$$\sim N(\underbrace{\underline{0}}_{\substack{\text{MEAN}\\\text{VECTOR}\\N\times 1}}, \underbrace{\underline{C}}_{\substack{\text{COVARIANCE}\\\text{MATRIX}\\N\times N}})$$

UNDER $\mathcal{H}_0$ $\quad \underline{x} \sim N(\underline{0}, \underline{C})$

UNDER $\mathcal{H}_1$ $\quad \underline{x} \sim N(\underline{s}, \underline{C})$

DECIDE $\mathcal{H}_1$ IF

$$l(\underline{x}) = \ln \frac{p(\underline{x}; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \ln \gamma.$$

$$l(\underline{x}) = \ln \frac{\dfrac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(\underline{C})} e^{-\frac{1}{2}(\underline{x}-\underline{s})^T \underline{C}^{-1}(\underline{x}-\underline{s})}}{\dfrac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(\underline{C})} e^{-\frac{1}{2}\underline{x}^T \underline{C}^{-1}\underline{x}}}$$

$$= \underline{x}^T \underline{C}^{-1} \underline{s} - \frac{1}{2} \underline{s}^T \underline{C}^{-1} \underline{s}$$

DECIDE $\mathcal{H}_1$ IF

$$T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s} > \underbrace{\ln \gamma + \frac{1}{2} \underline{s}^T \underline{C}^{-1} \underline{s}}_{\gamma'}$$

EXAMPLE : WGN $\underline{C} = \sigma^2 \underline{I}$

$$\Rightarrow T(\underline{x}) = \underline{x}^T \frac{1}{\sigma^2} \underline{I}\, \underline{s}$$

$$= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} x[n] s[n]$$

DECIDE $\mathcal{H}_1$ IF

$$\sum_n x[n]\, s[n] > \sigma^2 \gamma'$$

REPLICA-CORRELATOR

<u>EXAMPLE</u> : $w[n] \sim \mathcal{N}(0, \sigma_n^2)$

UNCORRELATED WITH UNEQUAL VARIANCES

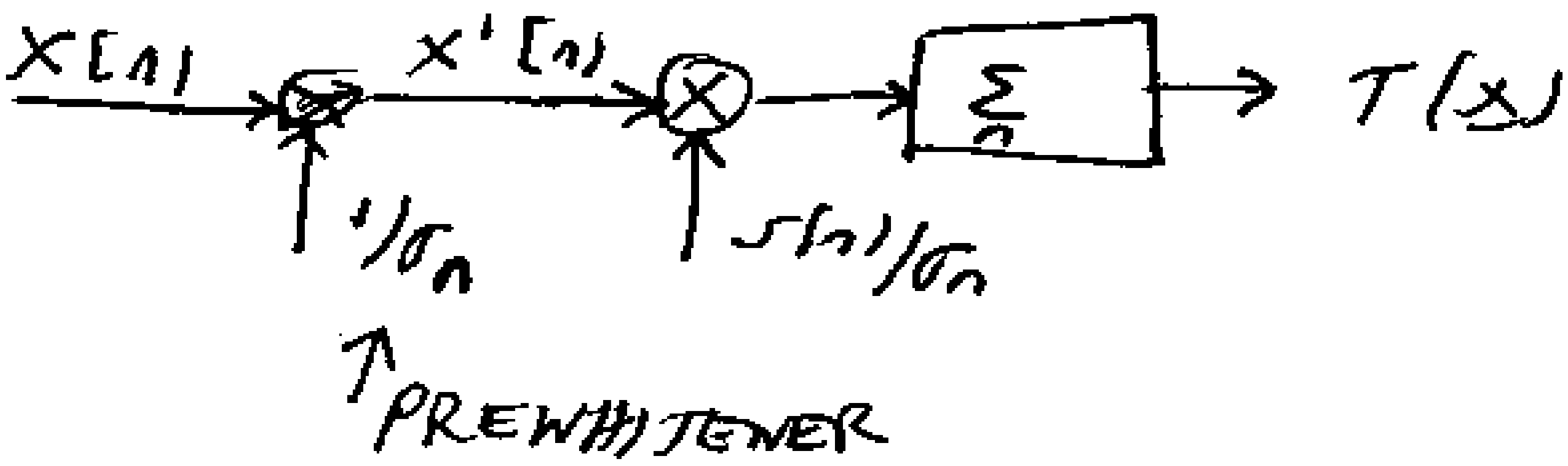
$$\underline{C} = \mathrm{DIAG}(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{N-1}^2)$$

$$\underline{C}^{-1} = \mathrm{DIAG}(1/\sigma_0^2, 1/\sigma_1^2, \ldots, 1/\sigma_{N-1}^2)$$

$$T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s}$$

$$= \sum_{n=0}^{N-1} \frac{x[n]\, s[n]}{\underbrace{\sigma_n^2}_{\text{WEIGHTS}}}$$

STATISTIC EMPHASIZES $x[n]s[n]$ FOR WHICH σ_n^2 IS SMALL.



OPTIMAL DETECTOR FIRST PREWHITENS NOISE SAMPLES

$$x'[n] = x[n]/\sigma_n$$

$$= w[n]/\sigma_n \qquad \mathcal{H}_0$$

$$\frac{w[n]}{\sigma_n} + \frac{s[n]}{\sigma_n} \qquad \mathcal{H}_1$$

BUT $\frac{w[0]}{\sigma_0}, \frac{w[1]}{\sigma_1}, \ldots, \frac{w[N-1]}{\sigma_{N-1}}$ IS WGN.

THEN $s'[n] = s[n]/\sigma_n$ IS USED AS THE REPLICA SINCE ORIGINAL SIGNAL IS ALSO PREWHITENED.

$T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s}$ CALLED A

GENERALIZED M.F. OR CORRELATOR

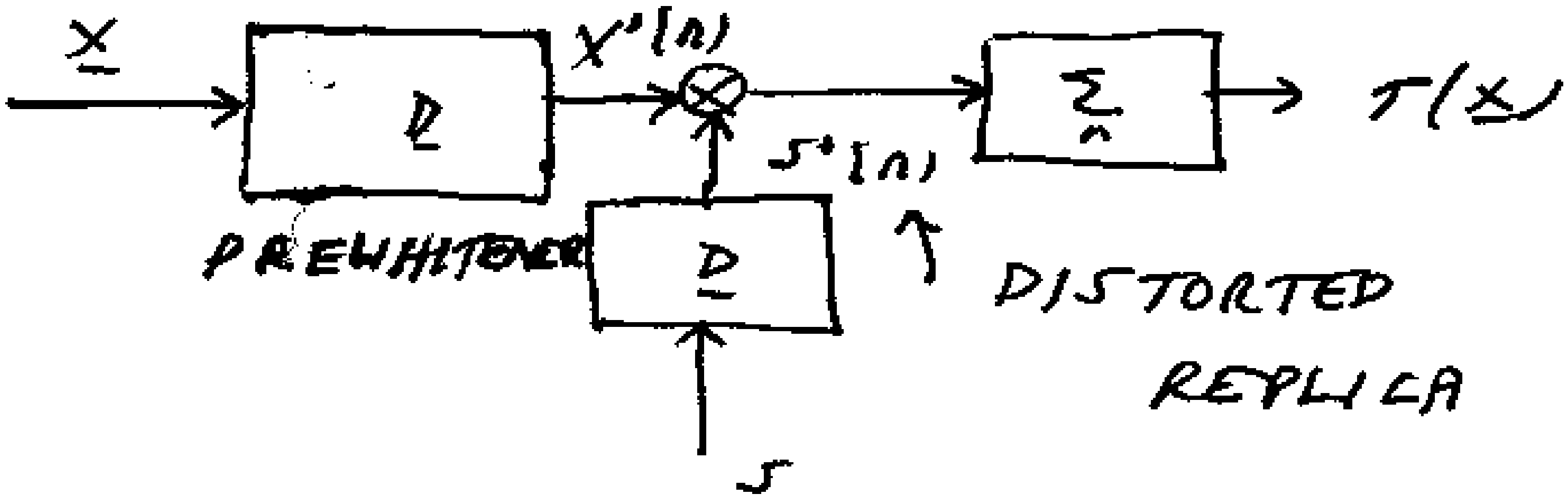
ALTERNATIVE STRUCTURE:

LET $\underline{C}^{-1} = \underline{D}^T \underline{D}$ $\qquad$ $\underline{D}$ IS NONSINGULAR (CHOLESKY FOR EXAMPLE)

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

$$= \mathbf{x}^T \mathbf{D}^T \mathbf{D} \mathbf{s}$$

$$= \mathbf{x}'^T \mathbf{s}'$$



NOISE AT OUTPUT OF $\mathbf{D}$ IS WGN SINCE

$$\mathbf{w}' = \mathbf{D}\mathbf{w} \qquad \mathbf{C}_{w'} = E(\mathbf{w}'\mathbf{w}'^T)$$

$$= E(\mathbf{D}\mathbf{w}\mathbf{w}^T\mathbf{D}^T)$$

$$= \mathbf{D}\mathbf{C}\mathbf{D}^T$$

$$= \mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T = \mathbf{I}$$

PERFORMANCE

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} > \gamma'$$

$T(\mathbf{x})$ IS GAUSSIAN SINCE

$$\mathbf{x} \sim N(\mathbf{0}, \mathbf{C}) \qquad \mathcal{H}_0$$

$$\sim N(\mathbf{s}, \mathbf{C}) \qquad \mathcal{H}_1$$

AND $T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s} = (\underline{s}^T \underline{C}^{-1})^T \underline{x}$

$= \underline{a}^T \underline{x}$ LINEAR TRANSFORMATION

$$E(T; \mathcal{H}_0) = E(\underline{w}^T \underline{C}^{-1} \underline{s}) = 0$$
$$E(T; \mathcal{H}_1) = E((\underline{s} + \underline{w})^T \underline{C}^{-1} \underline{s}) = \underline{s}^T \underline{C}^{-1} \underline{s}$$

$$\begin{aligned} VAR(T; \mathcal{H}_0) &= E(T^2) \\ &= E((\underline{w}^T \underline{C}^{-1} \underline{s})^2) \\ &= E(\underline{s}^T \underline{C}^{-1} \underline{w} \underline{w}^T \underline{C}^{-1} \underline{s}) \\ &= \underline{s}^T \underline{C}^{-1} E(\underline{w} \underline{w}^T) \underline{C}^{-1} \underline{s} \\ &= \underline{s}^T \underline{C}^{-1} \underline{s} \end{aligned}$$

SIMILARLY $VAR(T; \mathcal{H}_1) = \underline{s}^T \underline{C}^{-1} \underline{s}$

$T(\underline{x}) \sim N(0, \underline{s}^T \underline{C}^{-1} \underline{s}) \quad \mathcal{H}_0$

$N(\underline{s}^T \underline{C}^{-1} \underline{s}, \underline{s}^T \underline{C}^{-1} \underline{s}) \quad \mathcal{H}_1$

(GAUSS-GAUSS PROBLEM)

AS BEFORE

$$P_{FA} = Q(\gamma' / \sqrt{\underline{s}^T \underline{C}^{-1} \underline{s}})$$

$$P_D = Q\left(\frac{\gamma' - \underline{s}^T \underline{C}^{-1} \underline{s}}{\sqrt{\underline{s}^T \underline{C}^{-1} \underline{s}}} \right)$$

$$\Rightarrow \quad P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

$$d^2 = \underline{s}^T \underline{C}^{-1} \underline{s}$$

<u>EXAMPLE</u> : WGN $\quad \underline{C} = \sigma^2 \underline{I}$

$$d^2 = \frac{1}{\sigma^2} \underline{s}^T \underline{s} = \mathcal{E}/\sigma^2 = \eta$$

PERFORMANCE IMPROVES WITH $\underline{s}^T \underline{C}^{-1} \underline{s}$
(NOT SOLELY DEPENDENT ON $\mathcal{E} = \underline{s}^T \underline{s}$)

<u>EXAMPLE</u> : UNCORRELATED NOISE
WITH UNEQUAL VARIANCES

$$\underline{C} = \text{DIAG}\left(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{N-1}^2\right)$$

$$\underline{s}^T \underline{C}^{-1} \underline{s} = \sum_{n=0}^{N-1} \frac{s^2(n)}{\sigma_n^2}$$

FOR GIVEN σ_n^2's HOW CAN WE CHOOSE SIGNAL TO MAXIMIZE $\underline{s}^T \underline{C}^{-1} \underline{s}$ (SIGNAL DESIGN PROBLEM)?

MUST CONSTRAIN $\mathcal{E} = \underline{s}^T \underline{s}$ OR ELSE SOLUTION IS TO LET $s(n) \rightarrow \infty$.

MAXIMIZE $\underline{s}^T \underline{C}^{-1} \underline{s}$ WITH $\underline{s}^T \underline{s} = \mathcal{E}$

<u>SOLUTION</u> : USE LAGRANGIAN MULTIPLIERS

$$F = \sum_n \frac{s^2(n)}{\sigma_n^2} + \lambda\left(\mathcal{E} - \sum_n s^2[n]\right)$$

$$\frac{\partial F}{\partial s(k)} = \frac{2\, s(k)}{\sigma_k^2} - 2\lambda s[k] = 0$$

$$= 2\, s(k)\left(\frac{1}{\sigma_k^2} - \lambda\right)$$

ASSUMING $s(k) \neq 0 \Rightarrow \lambda = 1/\sigma_k^2$

IF ALL σ_n^2'S ARE DISTINCT $\lambda = 1/\sigma_k^2$ CAN HOLD FOR ONLY ONE k. FOR THE OTHERS $s(k) = 0$.

$$\Rightarrow \quad s(n) = \begin{cases} 0 & n \neq k \\ s(k) & n = k \end{cases}$$

$$\underline{s}^T \underline{C}^{-1} \underline{s} = \sum \frac{s^2(n)}{\sigma_n^2} = \frac{s^2(k)}{\sigma_k^2}$$

TO MAXIMIZE CHOOSE k FOR WHICH σ_k^2 IS <u>MINIMUM</u>.

THEN $s[n] = \sqrt{\mathcal{E}} \quad n = k$
$\qquad\qquad 0 \quad n \neq k$

CONCENTRATE ALL ENERGY AT SAMPLE WITH SMALLEST AMOUNT OF NOISE.

IN GENERAL, FOR ANY $\underline{C}$ CHOOSE $\underline{s}$ AS THE EIGENVECTOR OF $\underline{C}$ WITH SMALLEST EIGENVALUE OR

$$\underline{s} = \sqrt{\mathcal{E}}\,\underline{v}_{MIN}, \text{ FOR } \underline{C}\,\underline{v}_{MIN} = \lambda_{MIN}\,\underline{v}_{MIN}.$$

SEE EX 4.5.

MULTIPLE SIGNALS

BINARY CASE

$$\mathcal{H}_0 : x[n] = s_0[n] + w[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : x[n] = s_1[n] + w[n] \qquad n = 0, 1, \ldots, N-1$$

↑ WGN

$s_0[n], s_1[n]$ ARE DETERMINISTIC AND KNOWN ($s_0[n] = \cos 2\pi f_0 n$,
$s_1[n] = \cos(2\pi f_0 n + \pi)$
$\Rightarrow$ BPSK)