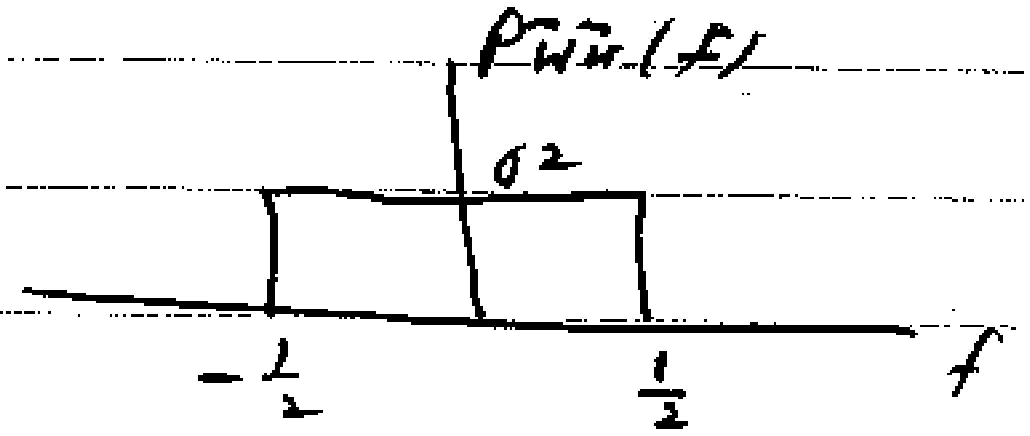
$$= E\left(W_R[n] W_R[n+k]\right) + E\left(W_I[n] W_I[n+k]\right)$$
$$= \sigma^2/2 \, \delta[k] + \sigma^2/2 \, \delta[k]$$
$$= \sigma^2 \delta[k]$$

$$P_{\tilde{w}\tilde{w}}(f) = \mathcal{F}\{ r_{\tilde{w}\tilde{w}}[k] \}$$
$$= \sigma^2 \qquad |f| \le \tfrac{1}{2}$$




TOTAL POWER $= \sigma^2$

## COMPLEX GAUSSIAN PDF

WE KNOW PDF OF $\{ W_R[0], \ldots, W_R[N-1], W_I[0], \ldots, W_I[N-1] \}$

SINCE THEY ARE IID WITH PDF $N(0, \sigma^2/2)$

<u>EXAMPLE</u>: $N = 1$

$$p\left(W_R[0], W_I[0]\right) = p\left(W_R[0]\right) p\left(W_I[0]\right)$$
$$= \frac{1}{\sqrt{2\pi \sigma^2/2}} e^{-\frac{1}{2\sigma^2/2} W_R^2[0]} \cdot \frac{1}{\sqrt{2\pi\sigma^2/2}} e^{-\frac{1}{2\sigma^2/2} W_I^2[0]}$$

$$= \frac{1}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}(w_R^2[0] + w_I^2[0])}$$

$$= \frac{1}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}|\tilde{w}[0]|^2}$$

COMPLEX NUMBERS

DEFINED FOR ALL $\tilde{w}[0] \in \mathbb{C}$

THIS PDF $p(w_R[0], w_I[0]) = p(\tilde{w}[0])$
CALLED THE <u>COMPLEX GAUSSIAN PDF</u>
OF THE COMPLEX RANDOM VARIABLE $\tilde{w}[0]$

ALSO WE SAY $\tilde{w}[0] \sim \mathcal{CN}(0, \sigma^2)$

NOTE

$$\mathrm{VAR}(\tilde{w}[0]) = E(|\tilde{w}[0]|^2) - |E(\tilde{w}[0])|^2$$

$$E(\tilde{w}[0]) = E(w_R[0] + j w_I[0])$$

$$= E(w_R[0]) + jE(w_I[0]) = 0$$

$$\mathrm{VAR}(\tilde{w}[0]) = E(|\tilde{w}[0]|^2) = \sigma^2$$

WHY?

---

FOR CWGN SHOW THAT

$$p(\tilde{w}[0], \ldots, \tilde{w}[N-1]) = \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\sum_{n=0}^{N-1}|\tilde{w}[n]|^2}$$

OR

$$p(\underline{\tilde{w}}) = \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\underline{\tilde{w}}^H \underline{\tilde{w}}} \qquad (H = *^T)$$

NOTE SIMILARITY TO REAL WGN PDF

$$p(\underline{w}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\underline{w}^T \underline{w}}$$

## COMPLEX DATA DETECTORS
## (CHAPTER 13)

WISH TO EXTEND MATCHED FILTER AND ESTIMATOR-CORRELATOR

## MATCHED FILTER

KNOWN COMPLEX DETERMINISTIC SIGNAL IN CWGN

$$\left.\begin{array}{ll} \mathcal{H}_0: & \tilde{x}[n] = \tilde{w}[n] \\ \mathcal{H}_1: & \tilde{x}[n] = \underbrace{\tilde{s}[n]}_{\text{KNOWN}} + \underbrace{\tilde{w}[n]}_{\text{CWGN} \sim \mathcal{CN}(0,\sigma^2)} \end{array}\right\} n = 0, 1, \ldots, N-1$$

USING NP AND GAUSSIAN PDF FOR CWGN

$$L(\tilde{\underline{x}}) = \frac{p(\tilde{\underline{x}}; \mathcal{H}_1)}{p(\tilde{\underline{x}}; \mathcal{H}_0)}$$

(RECALL THAT $p(\tilde{\underline{x}})$ IS REALLY
$p(\tilde{x}[0], \ldots, \tilde{x}[N-1]) = p(x_R[0], x_I[0], \ldots$
$x_R[N-1], x_I[N-1])$
USING COMPLEX NOTATION SIMPLIES ALGEBRA - NO NEW THEORY REQUIRED, CAN ALSO VIEW APPROACH AS MAPPING FROM $\begin{bmatrix} x \\ y \end{bmatrix}$ TO $x + jy$ OR $R^2 \to C$)

UNDER $\mathcal{H}_0$ $\tilde{\underline{x}} = \tilde{\underline{w}} \sim CN(\underline{0}, \sigma^2 I)$

$$p(\tilde{\underline{w}}) = \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{1}{\sigma^2} \tilde{\underline{w}}^H \tilde{\underline{w}}}$$

UNDER $\mathcal{H}_1$ $\tilde{\underline{x}} = \tilde{\underline{s}} + \tilde{\underline{w}} \sim CN(\tilde{\underline{s}}, \sigma^2 I)$

$$p(\tilde{\underline{x}}) = \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{1}{\sigma^2} (\tilde{\underline{x}} - \tilde{\underline{s}})^H (\tilde{\underline{x}} - \tilde{\underline{s}})}$$

$$\Rightarrow \ln L(\tilde{\underline{x}}) = -\frac{1}{\sigma^2} \left( (\tilde{\underline{x}} - \tilde{\underline{s}})^H (\tilde{\underline{x}} - \tilde{\underline{s}}) - \tilde{\underline{x}}^H \tilde{\underline{x}} \right)$$

$$= -\frac{1}{\sigma^2} \left( -\tilde{\underline{x}}^H \tilde{\underline{s}} - \tilde{\underline{s}}^H \tilde{\underline{x}} + \tilde{\underline{s}}^H \tilde{\underline{s}} \right)$$

$$= \frac{2}{\sigma^2} Re(\underline{\tilde{s}}^H \underline{\tilde{x}}) - \frac{1}{\sigma^2} \underline{\tilde{s}}^H \underline{\tilde{s}}$$

OR WE DECIDE $\mathcal{H}_1$ IF

$$T(\underline{\tilde{x}}) = Re(\underline{\tilde{s}}^H \underline{\tilde{x}})$$

$$= Re\left( \sum_{n=0}^{N-1} \tilde{x}[n] \tilde{s}^*[n] \right) > \gamma'$$

<u>EXAMPLE</u> : $\tilde{s}[n] = \tilde{A}$ COMPLEX DC LEVEL

$$T(\underline{\tilde{x}}) = Re\left( \sum_{n=0}^{N-1} \tilde{x}[n] \tilde{A}^* \right)$$

UNDER $\mathcal{H}_0$ $T(\underline{\tilde{x}}) = Re\left( \sum_n \tilde{w}[n] \tilde{A}^* \right)$

AND $E(T) = 0$ SHOW THIS.

UNDER $\mathcal{H}_1$ $T(\underline{\tilde{x}}) = Re\left( \sum_n (\tilde{A} + \tilde{w}[n]) \tilde{A}^* \right)$

$$= Re\left( \sum_n |\tilde{A}|^2 + \sum_n \tilde{w}[n] \tilde{A}^* \right)$$

$$= \underbrace{\sum_n |\tilde{A}|^2}_{\text{ENERGY} = \mathcal{E}} + Re\left( \sum_n \tilde{w}[n] \tilde{A}^* \right)$$

CAN SHOW THAT

$$P_{FA} = Q\left( \frac{\gamma'}{\sqrt{\sigma^2 \mathcal{E}/2}} \right) \qquad \mathcal{E} = \sum_{n=0}^{N-1} |\tilde{s}[n]|^2$$

$$P_D = Q\left(\frac{\gamma' - \mathcal{E}}{\sqrt{\sigma^2 \mathcal{E}/2}}\right)$$

OR $P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$

WHERE $d^2 = \dfrac{2\mathcal{E}}{\sigma^2}$

$d^2$ IS TWICE WHAT WE HAD FOR REAL CASE – DUE TO MEAN UNDER $\mathcal{H}_1$ IS REAL $\Rightarrow$ DISCARD IMAGINARY PART OF $\sum_n x[n]\tilde{s}^*[n]$ TO FIND $T(\tilde{\underline{x}})$

## ESTIMATOR- CORRELATOR

RECALL FOR CWGN

$$p(\tilde{\underline{x}}) = \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{1}{\sigma^2}\tilde{\underline{x}}^H\tilde{\underline{x}}}$$

AND $\underline{C}_{\tilde{x}} = E(\tilde{\underline{x}}\tilde{\underline{x}}^H) = \sigma^2 \underline{I}$

TO SEE THIS CONSIDER $N = 2$:

$$\underline{C}_{\tilde{x}} = E\left(\begin{bmatrix}\tilde{x}[0] \\ \tilde{x}[1]\end{bmatrix}[\tilde{x}^*[0], \tilde{x}^*[1]]\right)$$

$$= \begin{bmatrix} E(|\tilde{x}[0]|^2) & E(\tilde{x}[0]\tilde{x}^*[1]) \\ E(\tilde{x}[1]\tilde{x}^*[0]) & E(|\tilde{x}[1]|^2) \end{bmatrix}$$

ALREADY SHOWED $E(|\tilde{x}[n]|^2) = \mathrm{VAR}(\tilde{x}[n]) = \sigma^2$

$$E(\tilde{x}[0]\,\tilde{x}^*[1]) = E((x_R[0] + j x_I[0])(x_R[1] - j x_I[1]))$$

$$= E(x_R[0] x_R[1]) - j E(x_R[0] x_I[1]) + j E(x_I[0] x_R[1]) + E(x_I[0] x_I[1])$$

$= 0$ SINCE ALL RANDOM VARIABLES ARE INDEPENDENT

$$\Rightarrow p(\tilde{\mathbf{x}}) = \frac{1}{\pi^N \det(\mathbf{C}_{\tilde{x}})} e^{-\tilde{\mathbf{x}}^H \mathbf{C}_{\tilde{x}}^{-1} \tilde{\mathbf{x}}}$$

THIS IS CALLED THE COMPLEX <u>MULTIVARIATE</u> GAUSSIAN PDF AND WE USE $\tilde{\mathbf{x}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\tilde{x}})$

SEE KAY-I, 1993 FOR DETAILS

BACK TO DETECTION PROBLEM -

CONSIDER A COMPLEX RANDOM SIGNAL WITH COVARIANCE MATRIX $\mathbf{C}_{\tilde{s}}$ (AND ZERO MEAN) AND INDEPENDENT OF CWGN

$$\left.\begin{aligned} \mathcal{H}_0 &: \tilde{x}[n] = \tilde{w}[n] \\ \mathcal{H}_1 &: \tilde{x}[n] = \underbrace{\tilde{s}[n]}_{\text{RANDOM SIGNAL}} + \underbrace{\tilde{w}[n]}_{\text{CWGN}} \end{aligned}\right\} n = 0, 1, \ldots, N-1$$

OR $\tilde{\underline{x}} \sim \mathcal{CN}(\underline{0}, \sigma^2 \underline{I})$ UNDER $\mathcal{H}_0$

$\mathcal{CN}(\underline{0}, \underline{C}_{\tilde{s}} + \sigma^2 \underline{I})$ UNDER $\mathcal{H}_1$

NP DETECTOR DECIDES $\mathcal{H}_1$ IF

$$L(\tilde{\underline{x}}) = \frac{p(\tilde{\underline{x}}; \mathcal{H}_1)}{p(\tilde{\underline{x}}; \mathcal{H}_0)} > \gamma$$

WHERE

$$p(\tilde{\underline{x}}; \mathcal{H}_0) = \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{1}{\sigma^2} \tilde{\underline{x}}^H \tilde{\underline{x}}}$$

$$p(\tilde{\underline{x}}; \mathcal{H}_1) = \frac{1}{\pi^N \det(\underline{C}_{\tilde{s}} + \sigma^2 \underline{I})} \cdot e^{-\tilde{\underline{x}}^H (\underline{C}_{\tilde{s}} + \sigma^2 \underline{I})^{-1} \tilde{\underline{x}}}$$

RESULT: DECIDE $\mathcal{H}_1$ IF

$$T(\tilde{\underline{x}}) = \tilde{\underline{x}}^H \underbrace{\underline{C}_{\tilde{s}} (\underline{C}_{\tilde{s}} + \sigma^2 \underline{I})^{-1} \tilde{\underline{x}}}_{\hat{\tilde{\underline{s}}} \,=\, \text{MMSE ESTIMATOR OF } \tilde{\underline{s}}} > \gamma'$$

EXAMPLE: NONFLUCTUATING POINT TARGET

$$\tilde{s}(n) = \underset{\text{RANDOM}}{\tilde{A}} \; \underbrace{\tilde{h}(n)}_{\text{KNOWN}}$$

ASSUME $\tilde{A} \sim CN(0, \sigma_A^2)$

IF $\tilde{h}[n] = e^{j 2\pi f_D n}$

DOPPLER FREQ.

$\tilde{s}[n] = |\tilde{A}| e^{j(2\pi f_D n + \measuredangle \tilde{A})}$

RAYLEIGH RANDOM VAR.

UNIFORM RANDOM VAR.

INDEPENDENT

SINCE $|\tilde{A}| = \sqrt{A_R^2 + A_I^2}$

$\measuredangle \tilde{A} = \arctan \frac{A_I}{A_R}$

AND $A_R \sim N(0, \sigma_A^2/2)$
$A_I \sim N(0, \sigma_A^2/2)$ > INDEPENDENT

THIS IS STANDARD MODEL FOR COMPLEX ACTIVE SONAR/RADAR RETURN FROM NONFLUCTUATING POINT TARGET.

NO BANDWIDTH EXPANSION (EXCEPT DOPPLER SHIFT)

NO DELAY SPREAD

( $\tilde{h}(t, \tau) = \tilde{A}\, \delta(\tau)$

DOPPLER SPREAD

DELAY SPREAD

LINEAR TIME-VARYING RANDOM CHANNEL/ TARGET IMPULSE RESPONSE )

WHAT IS $\underline{C}_{\tilde{s}}$?

$$\tilde{\underline{s}} = \tilde{A}\,\tilde{\underline{h}}$$

$$\underline{C}_{\tilde{s}} = E(\tilde{\underline{s}}\,\tilde{\underline{s}}^H) = E(\tilde{A}\,\tilde{\underline{h}}\,\tilde{A}^*\,\tilde{\underline{h}}^H)$$

$$= E(|\tilde{A}|^2)\,\underline{h}\,\underline{h}^H = \sigma_A^2\,\tilde{\underline{h}}\,\tilde{\underline{h}}^H$$

THIS IS RANK ONE $\Rightarrow$ CAN USE WOODBURY TO INVERT $(\underline{C}_{\tilde{s}} + \sigma^2 \underline{I})$

RESULT : DECIDE $\mathcal{H}_1$ IF

$$T'(\tilde{\underline{x}}) = \left| \sum_{n=0}^{N-1} \tilde{x}(n)\,\tilde{h}^*(n) \right|^2 > \gamma''$$

<u>EXAMPLE</u> : $\tilde{h}(n) = e^{j2\pi f_D n}$

$$\Rightarrow \frac{T'(\tilde{\underline{x}})}{N} = \underbrace{\frac{1}{N} \left| \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi f_D n} \right|^2}_{\text{PERIODOGRAM}}$$

PERIODOGRAM
($f_D$ IS NOT REALLY KNOWN)

SEE BOOK FOR PERFORMANCE