1) GLRT
2) RAO

1) GLRT

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$$

TO FIND MLE $\hat{A}$:

$$p(\underline{x}; A, \mathcal{H}_1) = \prod_{n=0}^{N-1} p(x[n] - As[n])$$

MUST MAXIMIZE OVER A

EXAMPLE: LAPLACIAN

$$p(\underline{x}; A, \mathcal{H}_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|x[n] - As[n]|}$$

$$J(A) = \sum_{n=0}^{N-1} |x[n] - As[n]|$$

NO ANALYTICAL SOLUTION

$$2 \ln L_G(\underline{x}) = 2 \sum_{n=0}^{N-1} \ln \frac{p(x[n] - \hat{A}s[n])}{p(x[n])}$$

$$= 2 \max_{A} \sum_{n=0}^{N-1} \ln \frac{p(x[n]-As[n])}{p(x[n])}$$

CAN'T SIMPLIFY

RECALL THAT SINCE $\underline{\theta} = A$, $r=1$
AND

$$2 \ln L_G(\underline{x}) \overset{a}{\sim} \begin{cases} \chi_1^2 & \mathcal{H}_0 \\ \chi_1'^2(\lambda) & \mathcal{H}_1 \end{cases}$$

$$\lambda = A^2 I(A=0)$$

2) RAO (NO NUISANCE PARAMETERS)

$T_R(\underline{x}) =$

$$\left.\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}}\right|^T_{\underline{\theta}=\underline{\theta}_0} \underline{I}^{-1}(\underline{\theta}_0) \frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}}$$

$$= \frac{\left(\left.\frac{\partial \ln p(\underline{x};A)}{\partial A}\right|_{A=0}\right)^2}{I(A=0)}$$

NO MLES REQUIRED

BUT $\dfrac{\partial \ln p(\underline{x}; A, \mathcal{H}_1)}{\partial A} = \dfrac{\partial}{\partial A} \sum_{n=0}^{N-1}$

$\cdot \ln p(x(n) - As(n))$

$$= \sum_{n=0}^{N-1} \left. \frac{\frac{dp(w)}{dw}}{p(w)} \right|_{w = x(n) - As(n)} (-s(n))$$

$$= -\sum_{n=0}^{N-1} \frac{\frac{dp(x(n))}{dx(n)}}{p(x(n))} s(n) \quad \text{AT } A = 0$$

$$\Rightarrow T_R(\underline{x}) = \frac{\left(\sum_{n=0}^{N-1} \frac{-\frac{dp(x(n))}{dx(n)}}{p(x(n))} s(n) \right)^2}{I(A=0)}$$

SIMILAR TO WEAK SIGNAL NP EXCEPT FOR $(\ )^2$ $(-\infty < A < \infty)$ AND $I(A=0)$ $(T_R(\underline{x}) \overset{a}{\sim} \chi_1^2)$.

TO FIND $I(A=0)$:

SEE BOOK $\quad I(A=0) = i(A) \sum_{n=0}^{N-1} s^2(n)$

DOESN'T DEPEND ON A

EXAMPLE: LAPLACIAN NOISE, $s(n) = 1$

$$p(x(n)) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}\, |x(n)|}$$

$$-\frac{\frac{d\,p(x(n))}{dx(n)}}{p(x(n))} = -\frac{d \ln p(x(n))}{dx(n)}$$

$$= \sqrt{2/\sigma^2}\, \mathrm{SGN}(x(n))$$

ALSO, $$I(A=0) = i(A) \sum_{n=0}^{N-1} s^2(n)$$

$$= 2/\sigma^2\, N$$

$$T_R(\underline{x}) = \frac{\left(\sum_{n=0}^{N-1} \sqrt{2/\sigma^2}\, \mathrm{SGN}(x(n)) \right)^2}{2N/\sigma^2}$$

$$= \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \mathrm{SGN}(x(n)) \right)^2$$

$$T_R(\underline{x}) \overset{a}{\sim} \chi_1^2 \qquad \mathcal{H}_0$$

$$\chi_1'^2(\lambda) \qquad \mathcal{H}_1 \qquad \lambda = A^2 I(A=0)$$

$$= 2NA^2/\sigma^2$$

LINEAR MODEL

EXTENDS THEOREM 7.1 FOR NONGAUSSIAN NOISE (IID) AND REPLACES GLRT BY RAO TEST.

MODEL: $\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$ ← IID WITH $p(w[n])$

$\underline{H}$: $N \times p$ KNOWN

$\underline{\theta}$: $p \times 1$ UNKNOWN

$$\mathcal{H}_0 : \underline{\theta} = \underline{0}$$
$$\mathcal{H}_1 : \underline{\theta} \neq \underline{0}$$

RAO TEST DECIDES $\mathcal{H}_1$ IF

$$T_R(\underline{x}) = \frac{\underline{y}^T \underline{H} (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{y}}{i(A)} > \gamma'$$

WHERE $\underline{y} = [\,y[0]\; y[1] \ldots y[N-1]\,]^T$

$$y[n] = g(x[n])$$

$$g(w) = -\frac{dp(w)/dw}{p(w)}$$

$$i(A) = \int_{-\infty}^{\infty} \frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)} dw$$

PERFORMANCE (AS $N \to \infty$) IS

$$P_{FA} = Q_{\chi_p^2}(\gamma')$$

$$P_D = Q_{\chi_p'^2(\lambda)}(\gamma')$$

WHERE $\lambda = \dfrac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{1/i(A)}$ VALUE UNDER $\mathcal{H}_1$

SIGNAL PROCESSING EXAMPLE

SINUSOIDAL DETECTION IN IID NONGAUSSIAN NOISE (SEE 7.6.2 FOR WGN)

$$\mathcal{H}_0 : x(n) = w(n) \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : x(n) = A\cos(2\pi f_0 n + \phi) + w(n) \qquad ''$$

A, ϕ ARE UNKNOWN
f_0 KNOWN

$w[n]$ ARE IID WITH GENERALIZED GAUSSIAN PDF

CAN APPLY PREVIOUS THEOREM SINCE

$$A\cos(2\pi f_0 n + \phi) = \alpha_1 \cos 2\pi f_0 n + \alpha_2 \sin 2\pi f_0 n$$

$$\Rightarrow \quad \underline{\theta} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \qquad \underline{H} = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

NOW $\underline{H}^T\underline{H} \approx N/2\, \underline{I}$

$$T_R(\underline{x}) = \frac{\underline{y}^T \underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{y}}{1/i(A)}$$

$$= \frac{2}{N}\,\frac{\underline{y}^T\underline{H}\underline{H}^T\underline{y}}{1/i(A)}$$

$$= \frac{2\,i(A)}{N}\left[\left(\sum_n y(n)\cos 2\pi f_0 n\right)^2 + \left(\sum_n y(n)\sin 2\pi f_0 n\right)^2\right]$$

$$= 2\,i(A)\ \underbrace{\frac{1}{N}\left|\sum_{n=0}^{N-1} y(n)\, e^{-j2\pi f_0 n}\right|^2}_{\text{PERIODOGRAM OF } y(n)}$$

$$= 2\,i(A)\, I_y(f_0)$$

TO FIND $y(n)$:

$$g(w) = -\frac{d\ln p(w)}{dw}$$

$$p(w) = \frac{c_1(\beta)}{\sqrt{\sigma^2}}\, e^{-c_2(\beta)\left|\frac{w}{\sqrt{\sigma^2}}\right|^{\frac{2}{1+\beta}}}$$

$$g(w) = c_2(\beta)\,\frac{d}{dw}\left|\frac{w}{\sqrt{\sigma^2}}\right|^{\frac{2}{1+\beta}}$$

BUT
$$\frac{d}{dw}|w|^{\frac{2}{1+\beta}} = \begin{cases} \frac{d}{dw}(w)^{\frac{2}{1+\beta}} & w>0 \\ \frac{d}{dw}(-w)^{\frac{2}{1+\beta}} & w<0 \end{cases}$$

$$= \frac{2}{1+\beta} \, |w|^{\frac{1-\beta}{1+\beta}} \, \mathrm{SGN}(w)$$

$$\Rightarrow \quad y(n) = \frac{2\left[\dfrac{\Gamma\left(\frac{3}{2}(1+\beta)\right)}{\sigma^2\,\Gamma\left(\frac{1}{2}(1+\beta)\right)}\right]^{1/1+\beta}}{1+\beta}$$

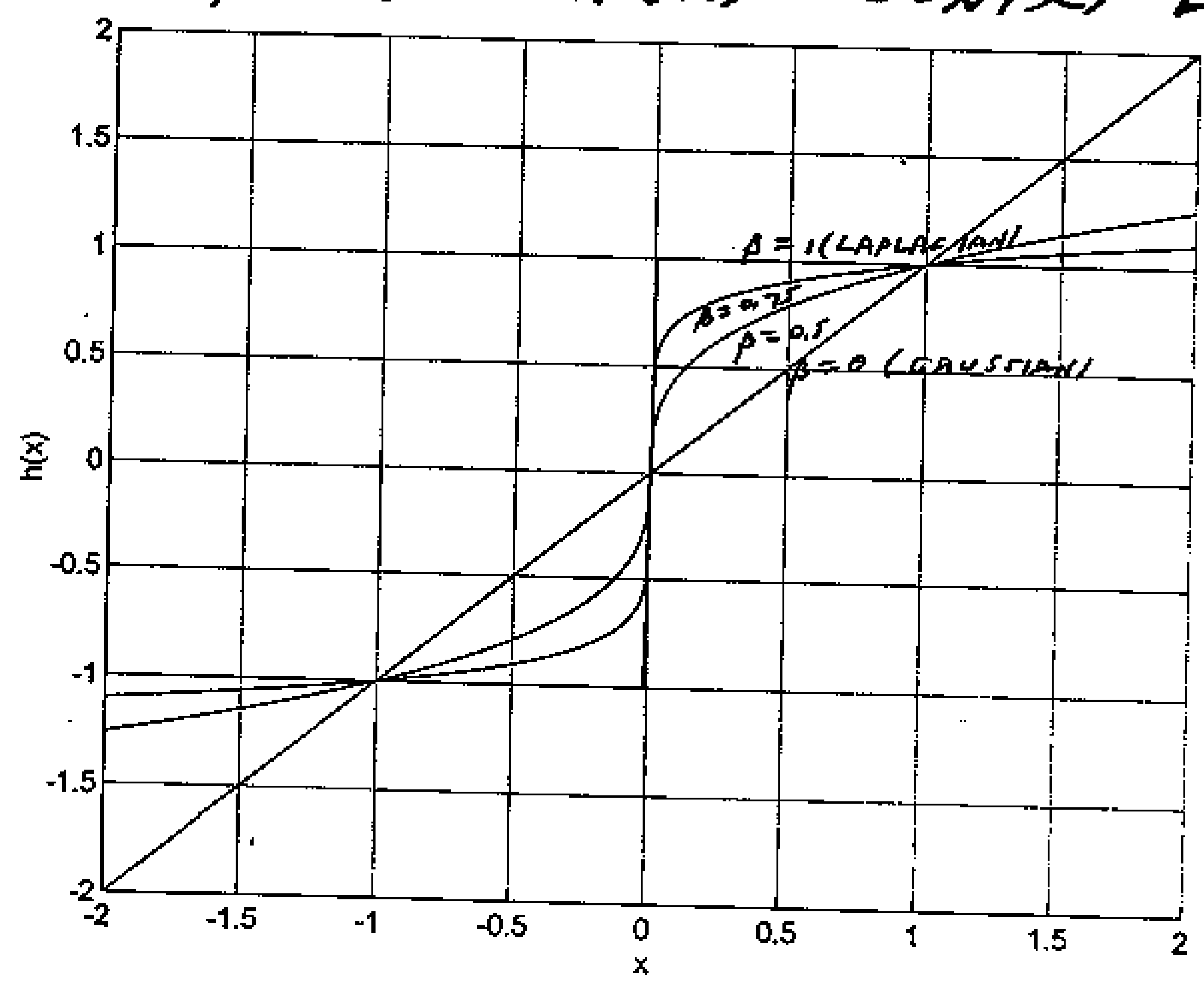
$$\cdot \; |x(n)|^{\frac{1-\beta}{1+\beta}} \, \mathrm{SGN}(x(n))$$

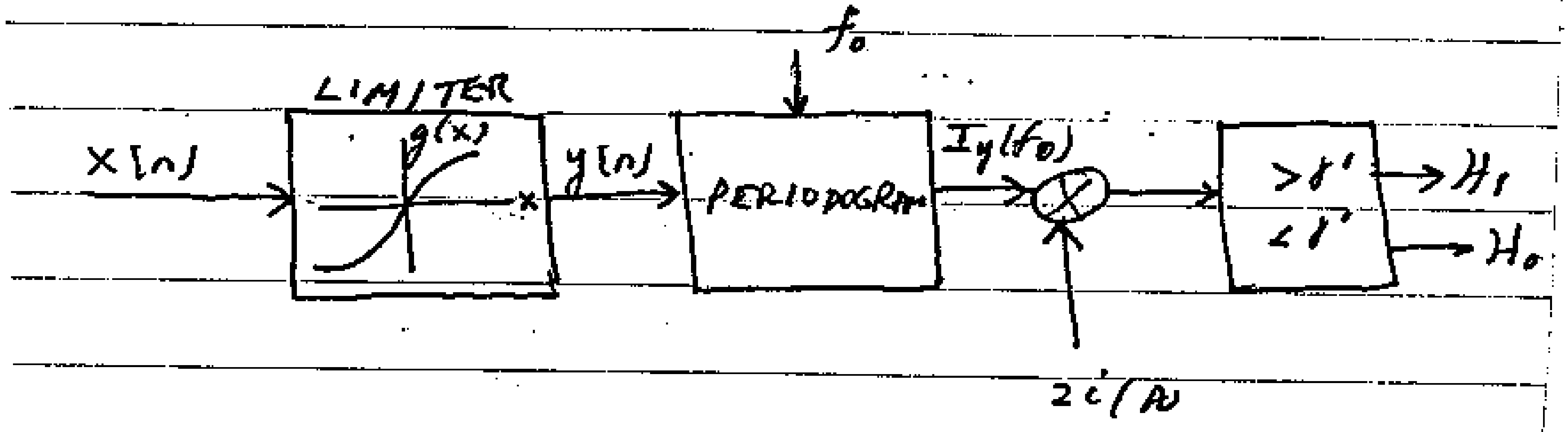
NORMALIZED NONLINEARITY IS

$$h(x) = |x|^{\frac{1-\beta}{1+\beta}} \, \mathrm{SGN}(x)$$

FOR $\beta = 0$ $\quad h(x) = x$ $\quad$ GAUSSIAN

$\beta = 1$ $\quad h(x) = \mathrm{SGN}(x)$ $\quad$ LAPLACIAN





ASYMPTOTIC PERFORMANCE FOLLOWS FROM THEOREM AS

$$P_{FA} = Q_{\chi^2_2}(\gamma') = e^{-\frac{1}{2}\gamma'}$$

$$P_D = Q_{\chi'^2_2(\lambda)}(\gamma')$$

$$\lambda = \frac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{1/i(A)}$$

$$= i(A) \left(\sum_{n=0}^{N-1} \alpha_1^2 \cos^2 2\pi f_0 n + \sum_{n=0}^{N-1} \alpha_2^2 \sin^2 2\pi f_0 n \right)$$

$$\approx i(A) \left(\frac{N}{2}\alpha_1^2 + \frac{N}{2}\alpha_2^2 \right) = \frac{N A^2 i(A)}{2}$$

EFFECT OF NONGAUSSIAN NOISE ON DETECTION PERFORMANCE IS SUMMARIZED BY $i(A)$ (INTRINSIC ACCURACY OF PDF)

GAUSSIAN NOISE $\Rightarrow$ $i(A) = 1/\sigma^2$

GENERALIZED GAUSSIAN NOISE
CAN SHOW THAT

$$i(A) = \frac{4/\sigma^2}{(1+\beta)^2} \frac{\Gamma\left(\frac{3}{2}(1+\beta)\right)\Gamma\left(\frac{3}{2}(1-\beta)\right)}{\Gamma^2\left(\frac{1}{2}(1+\beta)\right)}$$

$\geq 1/\sigma^2$ GAUSSIAN NOISE IS WORST CASE

GAIN IN PERFORMANCE =

$$10 \log_{10} \frac{i(A)}{1/\sigma^2} \text{ dB}$$

EASIER TO DETECT SIGNAL IN NONGAUSSIAN NOISE OF <u>SAME</u> <u>VARIANCE</u>

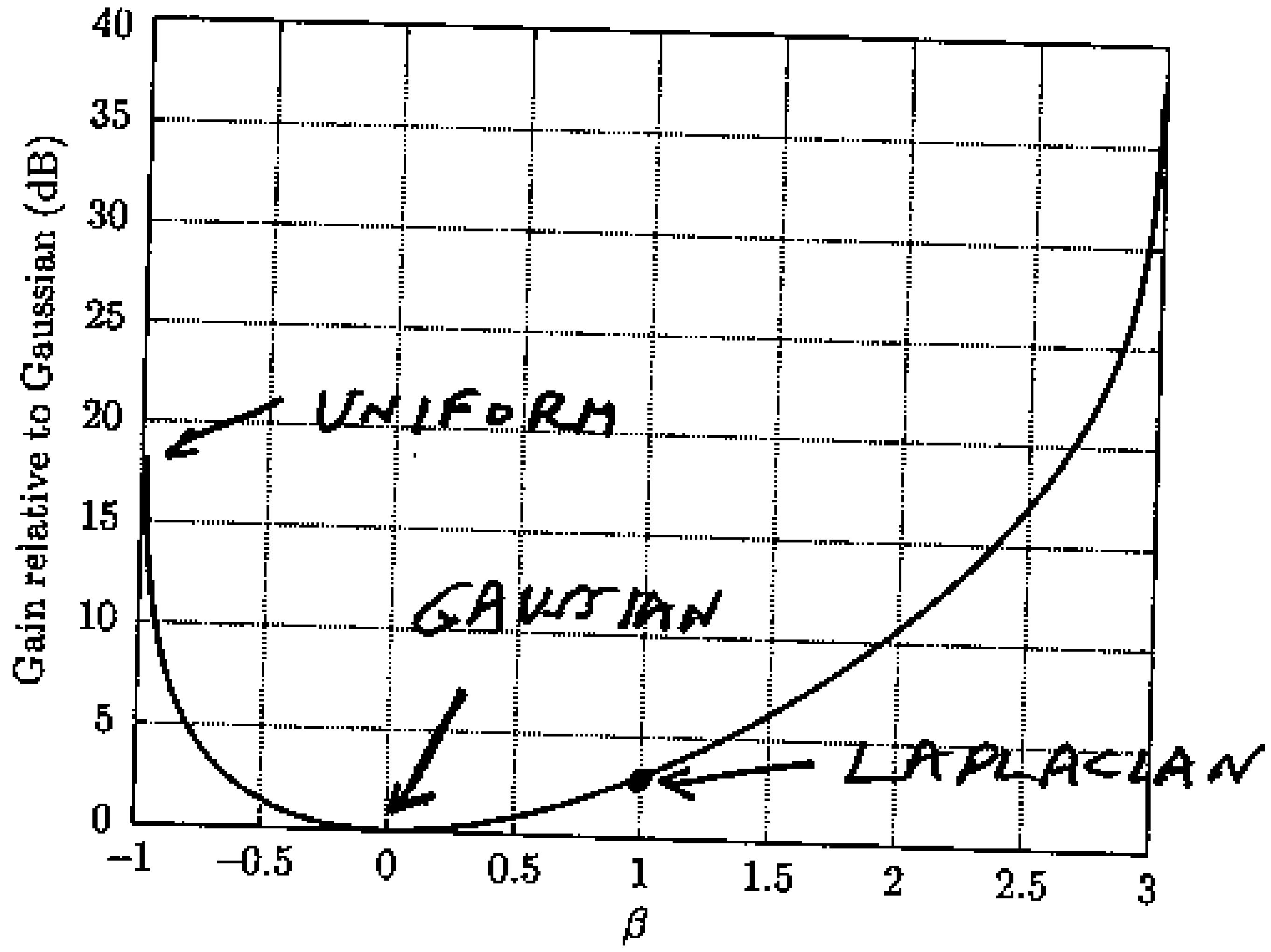


Figure 10.10. Asymptotic gain of nonGaussian PDF designed detector to linear detector.

LINEAR DETECTOR CORRESPONDS TO $g(x) = x/\sigma^2$ (ASSUMED $\beta = 0$)

REASON : LAPLACIAN IS LESS CONCENTRATED AT $W=0$ DUE TO LARGER TAILS $\Rightarrow$ SLIGHT SHIFT IS MORE NOTICEABLE

FINALLY, IF f_0 IS UNKNOWN, A REASONABLE APPROACH IS TO DECIDE $\mathcal{H}_1$ IF

$$\max_{f_0} T_R(\underline{x}) > \gamma'$$

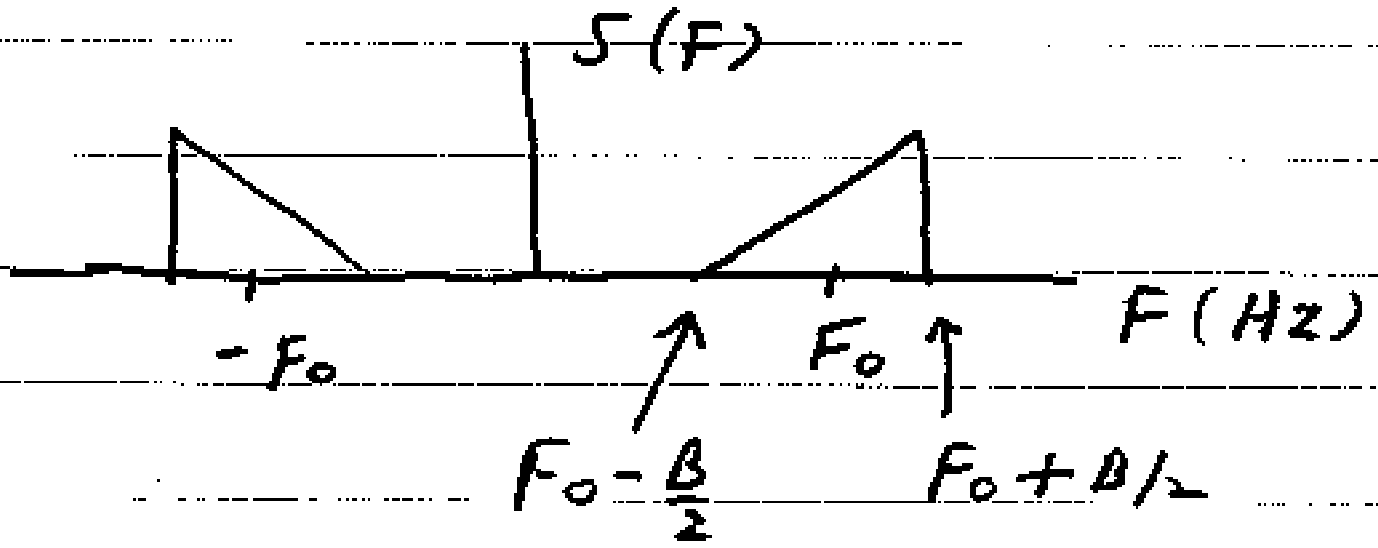
OR $$\max_{f_0} 2c'(A) I_y(f_0) > \gamma'$$

SINCE $T_R(\underline{x}) \simeq 2 \ln L_G(\underline{x})$ AS $N \to \infty$, AND GLRT WOULD MAXIMIZE OVER f_0.

DETECTION WITH COMPLEX DATA

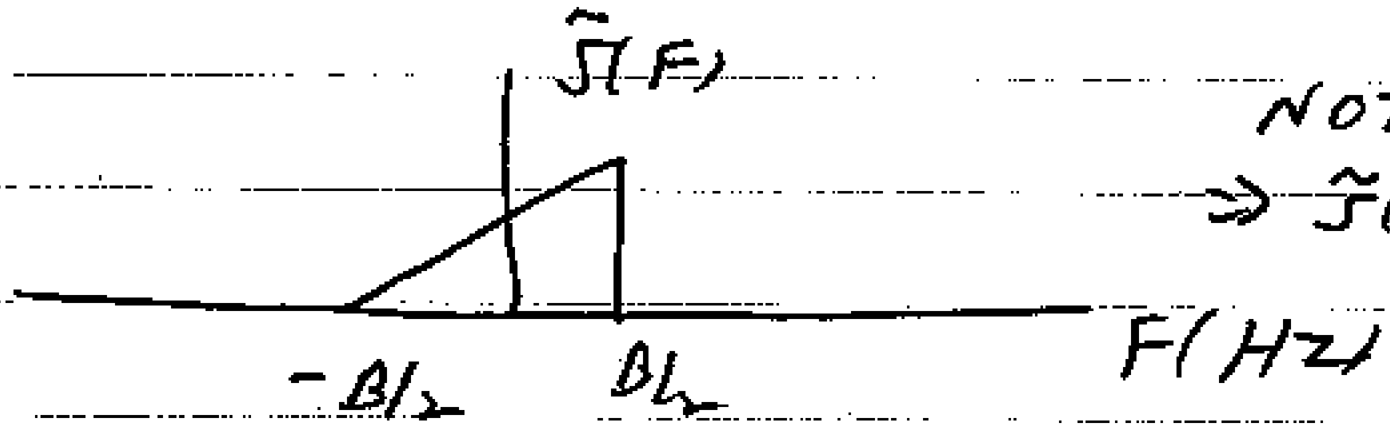
SEE KAY-I, 1993, CHAPTER 15

CONSIDER A BANDPASS SIGNAL $s(t)$



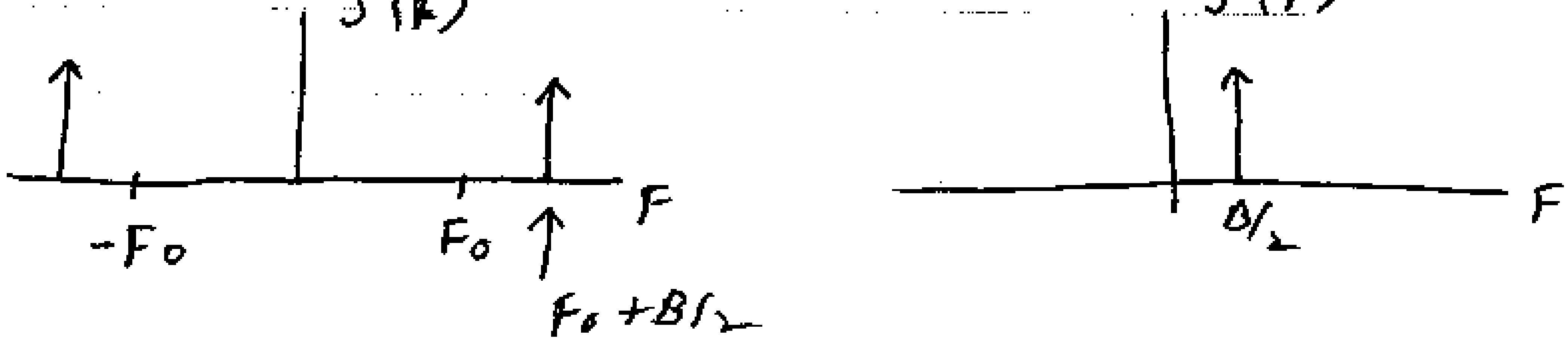
INSTEAD OF SAMPLING AT $F_s > 2(F_0 + B/2)$
WE DEMODULATE TO BASEBAND AND
SAMPLE AT LOWER RATE

BASEBAND SPECTRUM IS



NOT HERMITIAN
$\Rightarrow \tilde{s}(t)$ COMPLEX

EXAMPLE : $s(t) = A\cos 2\pi (F_0 + B/2)\, t$



TO FIND $\tilde{s}(t)$ NOTE THAT

$$S(F) = \tilde{S}(F - F_0) + \tilde{S}^*(-(F + F_0))$$

$$\Rightarrow s(t) = \tilde{s}(t) e^{j2\pi F_0 t} + \left[\tilde{s}(t) e^{j2\pi F_0 t}\right]^*$$

WHERE $\tilde{s}(t) = \mathcal{F}^{-1}\{\tilde{S}(F)\}$

OR $s(t) = 2\,\mathrm{Re}\left[\tilde{s}(t) e^{j2\pi F_0 t}\right]$

EXAMPLE : $s(t) = A\cos 2\pi(F_0 + B/2)t$

$$s(t) = 2\,\mathrm{Re}\Big[\underbrace{\frac{A}{2} e^{j2\pi B/2\, t}}_{\tilde{s}(t)\text{ - COMPLEX ENVELOPE}} e^{j2\pi F_0 t}\Big]$$

BANDWIDTH OF $\tilde{s}(t)$ is $B/2$ $\Rightarrow$
SAMPLE COMPLEX SIGNAL
AT $> B$ COMPLEX SAMPLES/SEC.

ALSO, IF $\tilde{s}(t) = s_R(t) + j s_I(t)$

$$s(t) = 2\,\mathrm{Re}\left[(s_R(t) + j s_I(t)) e^{j2\pi F_0 t}\right]$$

$$= 2 s_R(t)\cos 2\pi F_0 t - 2 s_I(t)\sin 2\pi F_0 t$$

IN PRACTICE WE OBTAIN $s_R(t)$, $s_I(t)$ AS FOLLOWS:

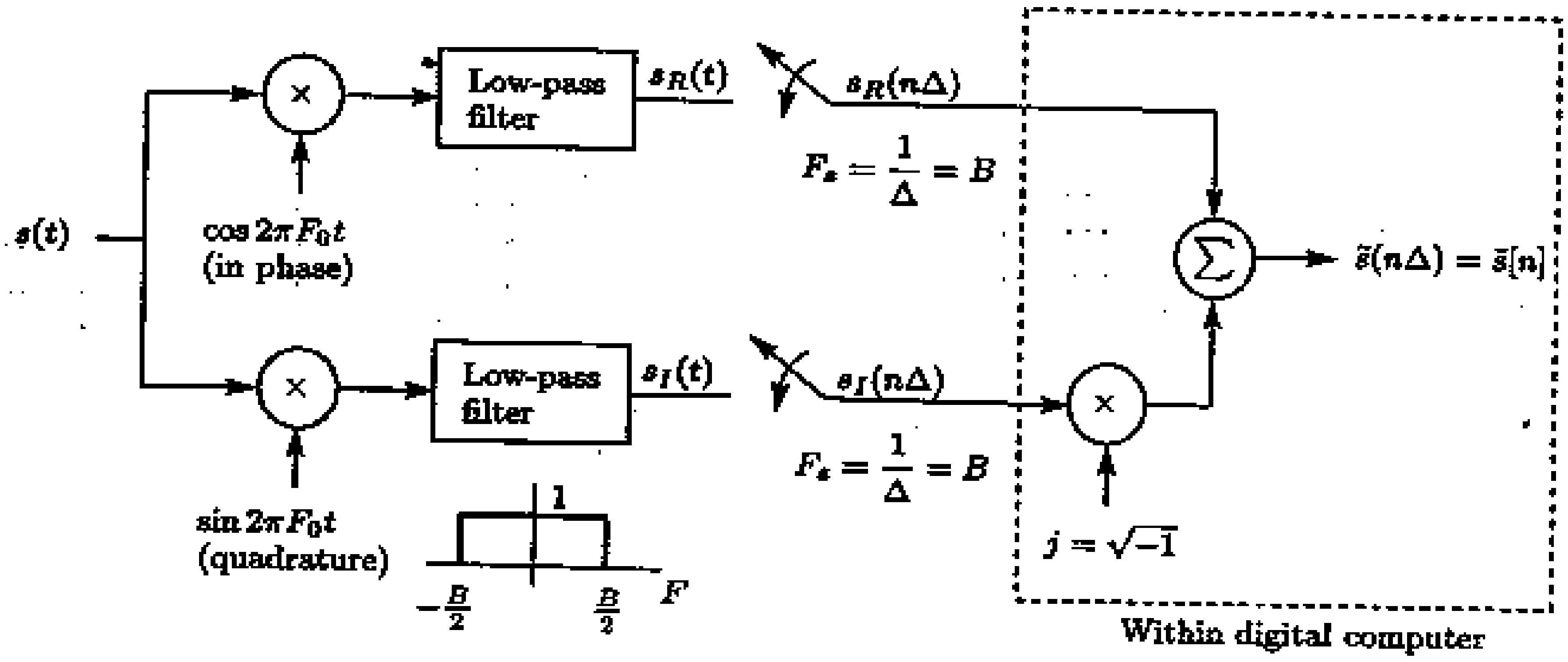


Figure 15.2 Extraction of discrete-time complex envelope from bandpass signal

IF WHITE GAUSSIAN NOISE IS PRESENT OR $x(t) = s(t) + w(t)$

$$P_w(F) = N_0/2 \qquad \text{ALL } F$$

WE HAVE INSIDE COMPUTER

$$\tilde{x}[n] = \tilde{s}[n] + \tilde{w}[n]$$

COMPLEX ENVELOPE (DISCRETE)

PROPERTIES ?

$$\tilde{w}[n] = \tilde{w}(n\Delta) \qquad F_s = 1/\Delta = B$$

$$= w_R(n\Delta) + j\, w_I(n\Delta)$$

WHERE

$$w_R(t) = \left[w(t) \cos 2\pi F_0 t \right]_{LPF}$$

$$w_I(t) = \left[w(t) \sin 2\pi F_0 t \right]_{LPF}$$

WE NEXT SHOW THAT ALL $w_R(n\Delta)$, $w_I(n\Delta)$ ARE GAUSSIAN, ZERO MEAN, VARIANCE $= \sigma^2/2$, AND INDEPENDENT.

PROOF: $w_R(t)$, $w_I(t)$ ARE JOINTLY GAUSSIAN (LPF IS LINEAR TRANSFORMATION) $\Rightarrow$ $w_R(n\Delta)$, $w_I(n\Delta)$ ARE JOINTLY GAUSSIAN

ALSO, $w_R(t)$, $w_I(t)$ ARE ZERO MEAN (WHY?) $\Rightarrow$ $w_R(n\Delta)$, $w_I(n\Delta)$ ARE ZERO MEAN.

TO FIND COVARIANCES: AS EXAMPLE

$$E\left(w_R(t) w_R(t')\right) = ?$$

$$w_R(t) = \int_{-\infty}^{\infty} w(u) \cos 2\pi F_0 u \underbrace{h(t-u)}_{LPF} du$$

$$w_R(t') = \int_{-\infty}^{\infty} w(v) \cos 2\pi F_0 v \, h(t'-v) \, dv$$

$E(W_R(t)W_R(t')) =$

$$\iint E(W(u)W(v)) \cos 2\pi F_0 u \cos 2\pi F_0 v \; h(t-u)\, h(t'-v)\, du\, dv$$

BUT $E(W(u)W(v)) = N_0/2 \; \delta(u-v)$

$$E(W_R(t)W_R(t')) = \frac{N_0}{2} \int_{-\infty}^{\infty} \cos^2 2\pi F_0 u \; h(t-u)\, h(t'-u)\, du$$

$$= \frac{N_0}{2} \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi F_0 u\right) h(t-u)\, h'(t-u)\, du$$

BUT $\int \cos 4\pi F_0 u \; h(t-u)\, h(t'-u)\, du$

$$= Re\Big[\underbrace{\int h(t-u)\, h'(t-u)\, e^{-j2\pi 2F_0 u}\, du}_{\text{FOURIER TRANSFORM AT } f = 2F_0 > B \Rightarrow \text{ MUST EQUAL ZERO}} \Big]$$

ALSO

$$\int_{-\infty}^{\infty} h(t-u)\, h(t'-u)\, du$$

$$= \int_{-\infty}^{\infty} \mathcal{F}\{h(t-u)\}\, \mathcal{F}^{*}\{h(t'-u)\}\, df$$

$$= \int_{-\infty}^{\infty} \left(H(f)\, e^{-j2\pi F t}\right)^{*} \left(H(f)\, e^{-j2\pi F t'}\right) dF$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 e^{j2\pi F(t-t')} dF$$

$$= \int_{-B/2}^{B/2} 1 \cdot e^{j2\pi F(t-t')} dF$$

$$= \left. \frac{e^{j2\pi F(t-t')}}{j2\pi(t-t')} \right|_{-B/2}^{B/2}$$

$$= \frac{e^{j\pi B(t-t')} - e^{-j\pi B(t-t')}}{j2\pi(t-t')}$$

$$= \frac{B \sin \pi B(t-t')}{\pi B(t-t')}$$

OR $E(W_R(t) W_R(t')) = \frac{N_0 B}{4} \frac{\sin \pi B(t-t')}{\pi B(t-t')}$

BUT $t = n\Delta$, $t' = m\Delta$

$= \frac{n}{B}$ $\quad = \frac{m}{B}$

$$E(W_R(n\Delta) W_R(m\Delta)) = \frac{N_0 B}{4} \frac{\sin \pi(n-m)}{\pi(n-m)}$$

$$= \frac{N_0 B}{4} \quad m = n$$

$$0 \quad m \neq n$$

OR $E(W_R(n) W_R(m)) = \frac{N_0 B}{4} \delta(n-m)$

SIMILARLY,

$$E\left(w_R[n]\,w_I[m]\right) = 0 \qquad \text{ALL } m, n$$

$$E\left(w_I[n]\,w_I[m]\right) = \frac{N_0 B}{4}\,\delta[n-m]$$

SUMMARY: COMPLEX ENVELOPE OF NOISE IS

$$\tilde{w}[n] = \underbrace{w_R[n]}_{\substack{\text{WGN} \\ \text{VAR} = \frac{N_0 B}{4} \\ = \sigma^2/2}} + j\,\underbrace{w_I[n]}_{\substack{\text{WGN} \\ \text{VAR} = \frac{N_0 B}{4} \\ = \sigma^2/2}}$$

AND $w_R[n]$, $w_I[n]$ ARE INDEPENDENT PROCESSES

$w[n]$ CALLED COMPLEX WGN (CWGN)

AND NOTE THAT

$$r_{\tilde{w}\tilde{w}}[k] = E\left[\tilde{w}^*[n]\,\tilde{w}[n+k]\right)$$

$$= E\left[\left(w_R[n] - j w_I[n]\right)\left(w_R[n+k] + j w_I[n+k]\right)\right]$$