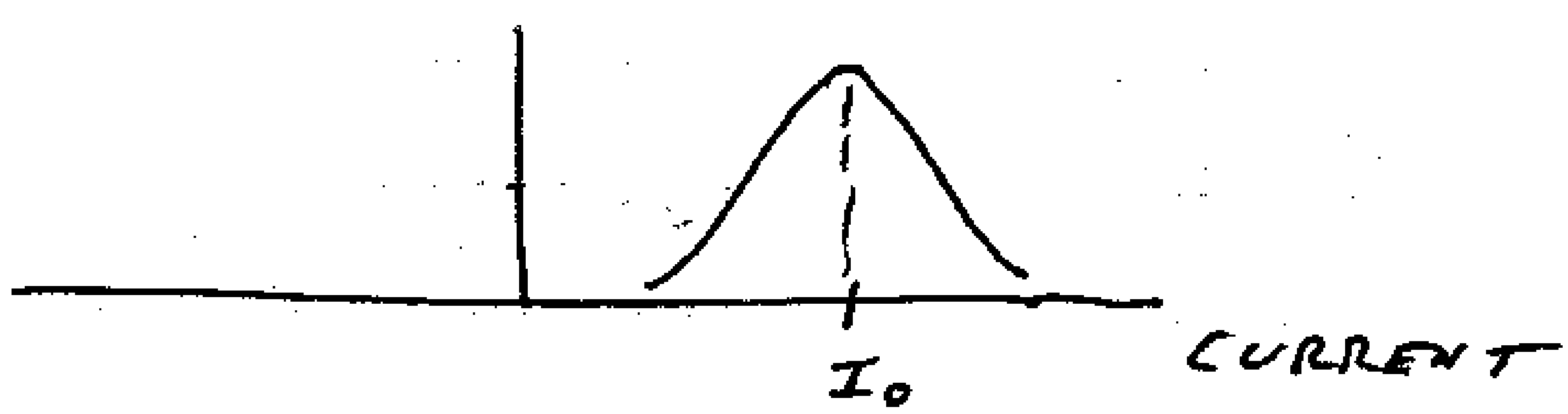
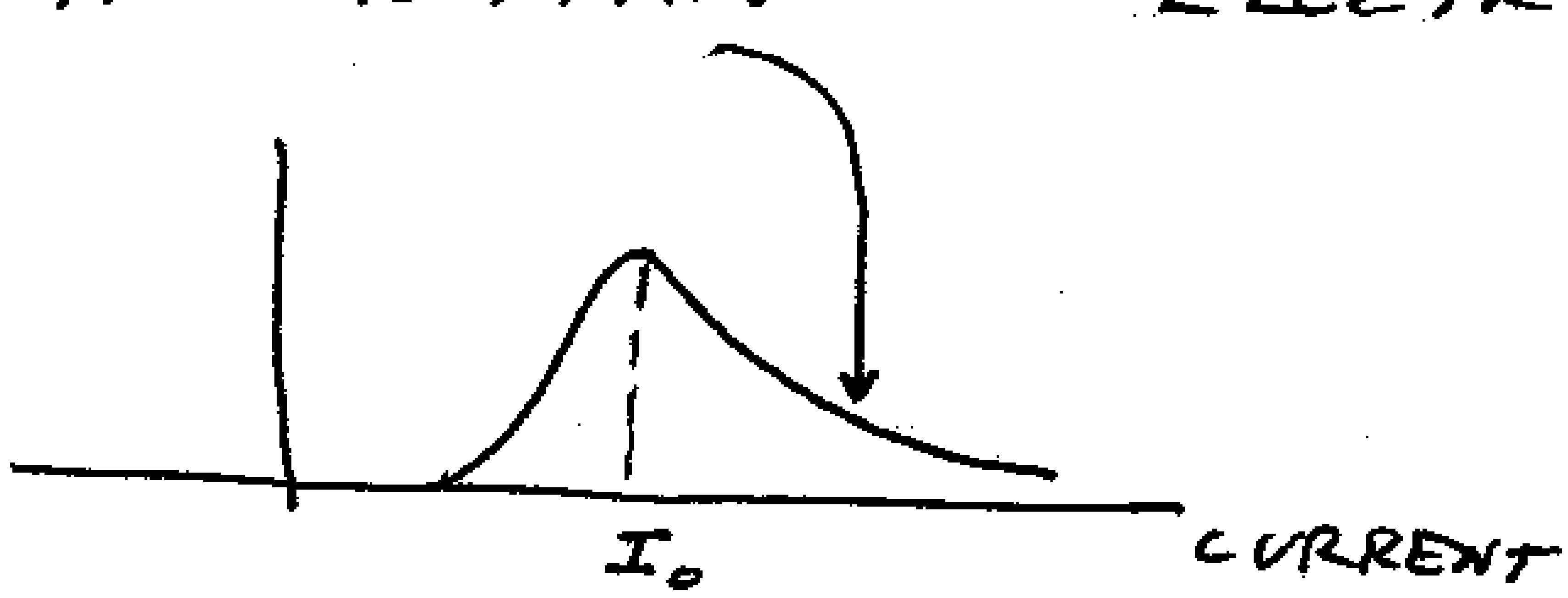
NONGAUSSIAN NOISE
(CHAPTER 10)

GAUSSIAN PDF MODEL JUSTIFIED BY CENTRAL LIMIT THEOREM

EXAMPLE : RESISTOR NOISE, ELECTRONS ALL MOVING AT ABOUT SAME VELOCITY ⇒ CURRENT OR CHARGE PER UNIT TIME IS SOME AVERAGE WITH SMALL DEVIATIONS



WHAT HAPPENS IF WE HAVE SOME " HOT RODDING " ELECTRONS?



NOISE IS NONGAUSSIAN.

OTHER EXAMPLES - HIGH LEVEL BUT INFREQUENT EVENTS SUCH AS ELECTROMAGNETIC NOISE SPIKES DUE TO THUNDERSTORMS OR ACOUSTIC SPIKES DUE TO ICEBERG BREAKUP

CONSIDER NOW DETERMINISTIC SIGNALS IN NONGAUSSIAN NOISE.

NOISE CHARACTERISTICS

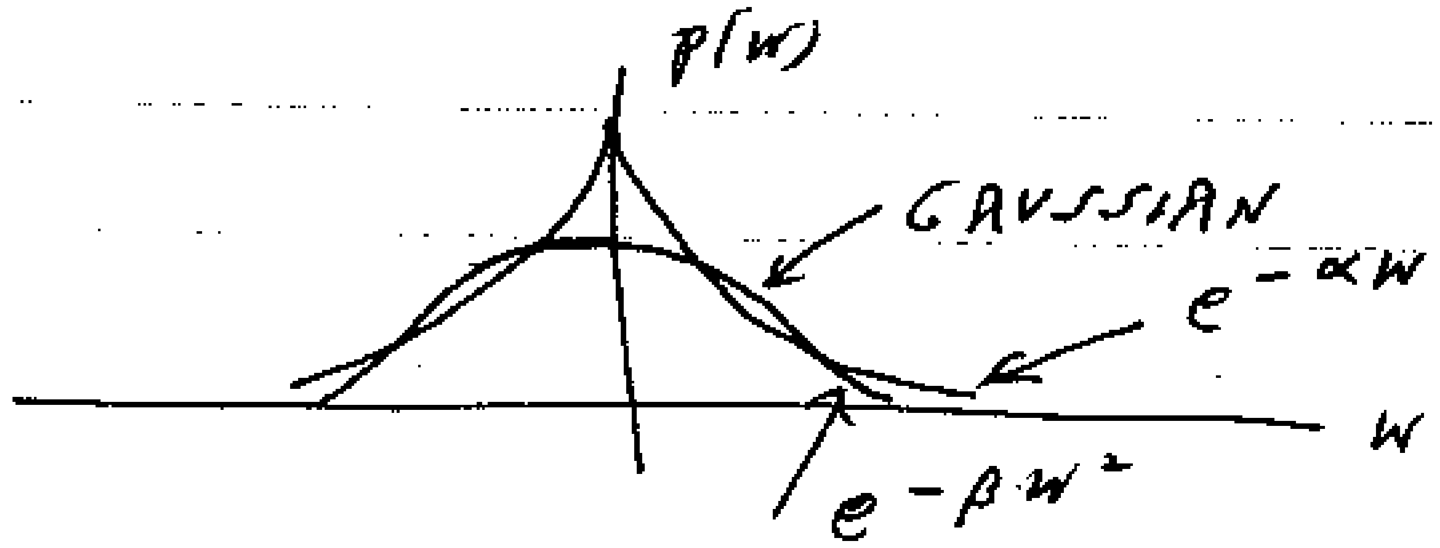
SIMPLEST MODEL IS IID

EXAMPLE : LAPLACIAN

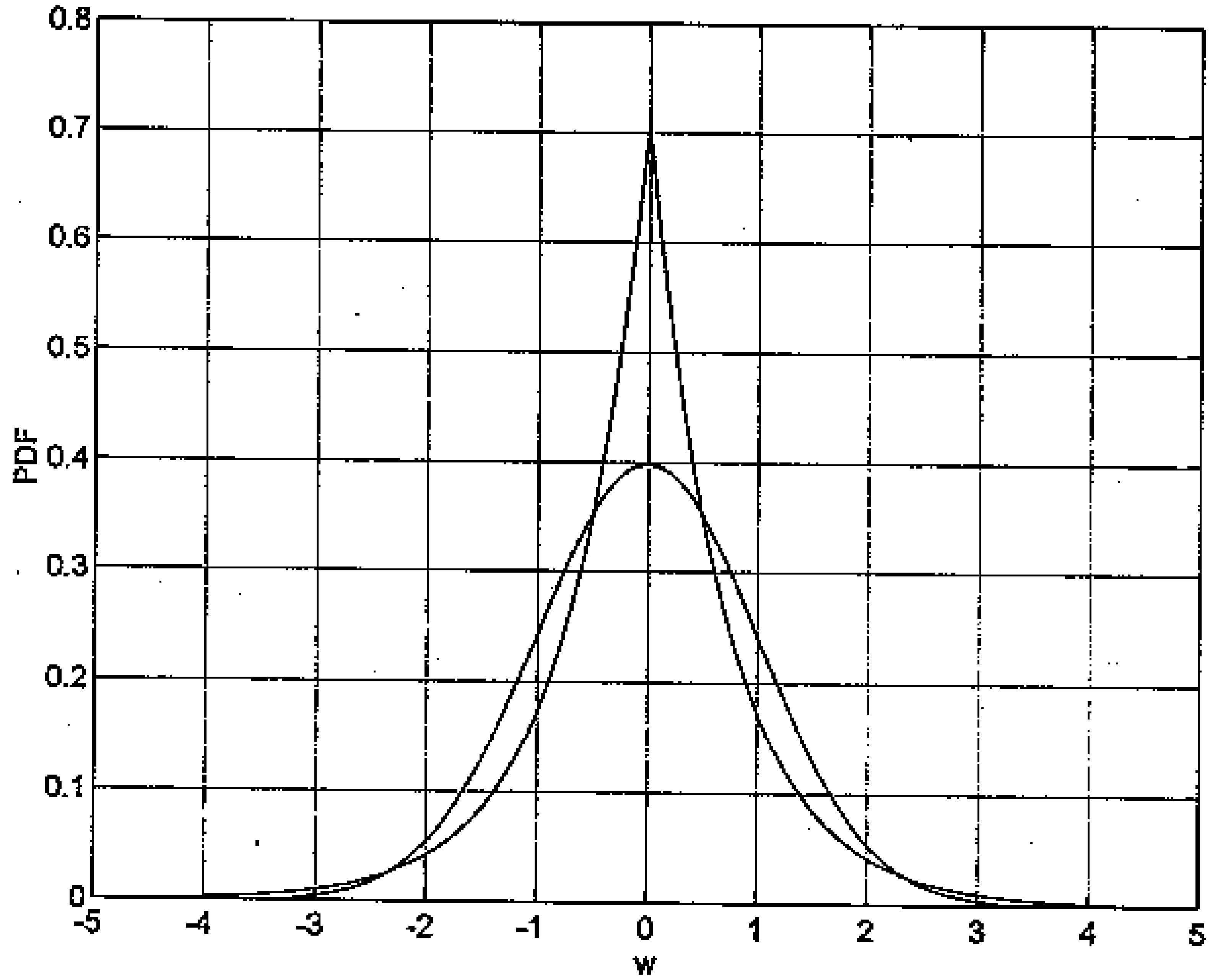
$$p(w[n]) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|w[n]|}$$

$$n = 0, 1, \ldots, N-1$$

ALL $w[n]$ HAVE SAME PDF AND ARE INDEPENDENT.



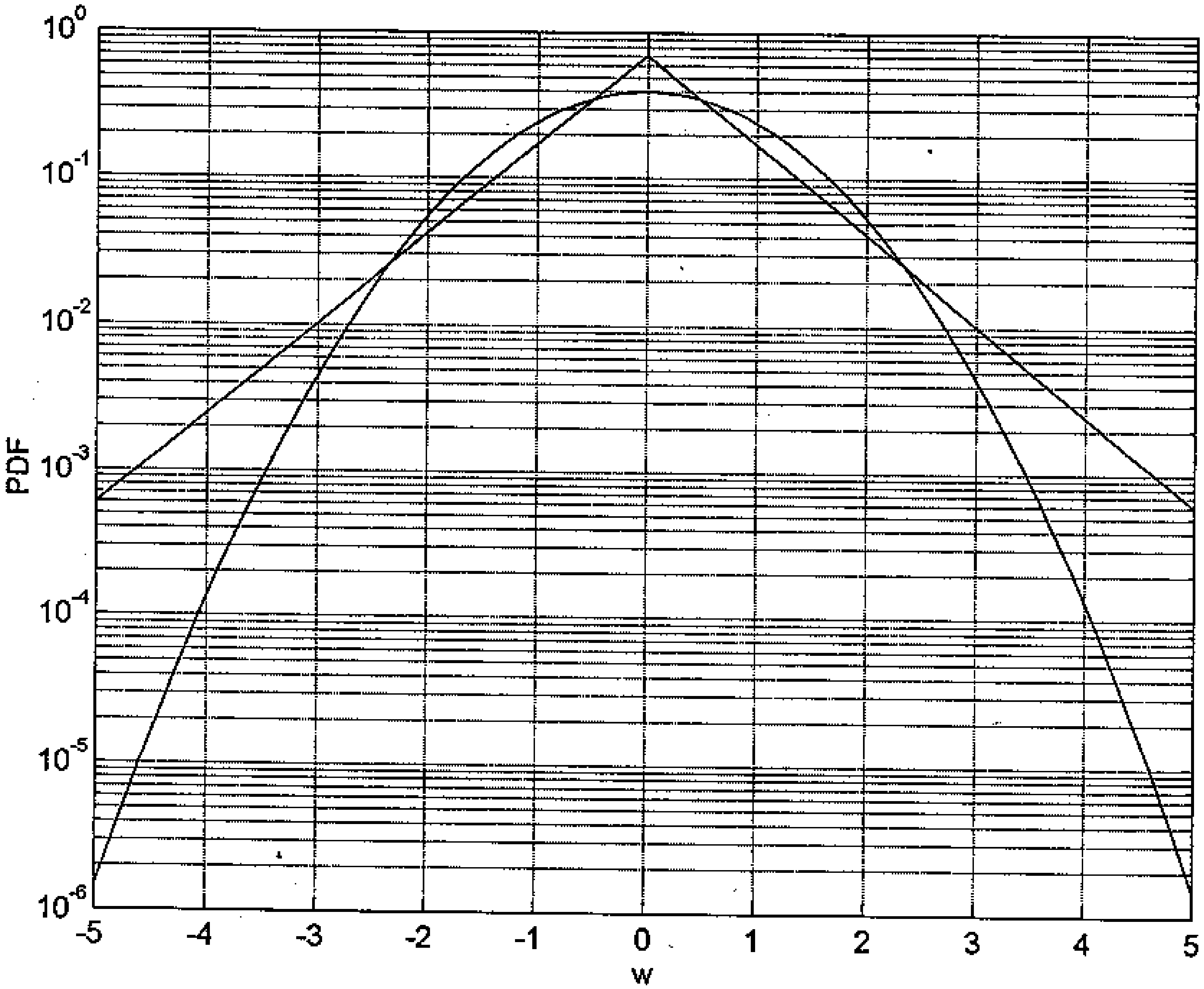
LAPLACIAN PDF "TAILS" ARE HEAVIER ⇒ MORE HIGH LEVEL EVENTS



a)

FIGURE 10.1 – GAUSSIAN VERSUS TYPICAL NONGAUSSIAN PDF ($\sigma^2 = 1$)

NOTE: BOTH PDFs HAVE SAME VARIANCE σ^2 – OTHERWISE COMPARING "APPLES AND ORANGES"



b)

FIGURE 10.1

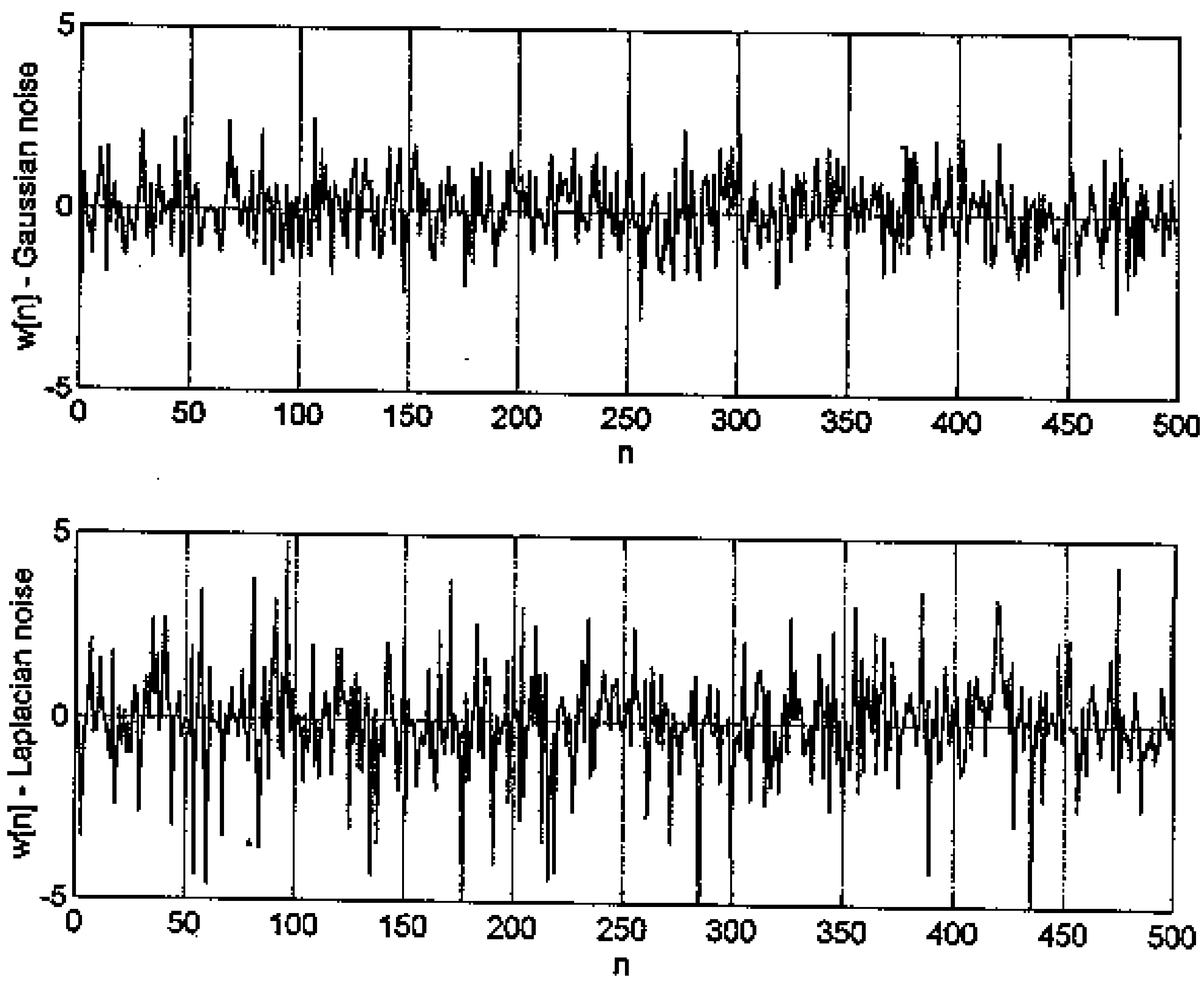


FIGURE 10.2 - REALIZATION OF GAUSSIAN AND NONGAUSSIAN NOISE PROCESSES ($\sigma^2 = 1$)

HIGH LEVEL EVENTS CALLED SPIKES OR OUTLIERS - CAN INCREASE P_{FA} FOR GAUSSIAN DESIGNED DETECTORS (NEED LIMITERS OR CLIPPERS)

GENERALIZED GAUSSIAN PDFS

ENCOMPASSES LAPLACIAN, GAUSSIAN, UNIFORM

$$p(w) = \frac{c_1(\beta)}{\sqrt{\sigma^2}} e^{-c_2(\beta)\left|\frac{w}{\sqrt{\sigma^2}}\right|^{\frac{2}{1+\beta}}} \qquad -\infty < w < \infty$$

WHERE
$$c_1(\beta) = \frac{\Gamma^{\frac{1}{2}}\left(\frac{3}{2}(1+\beta)\right)}{(1+\beta)\,\Gamma^{3/2}\left(\frac{1}{2}(1+\beta)\right)}$$

$$c_2(\beta) = \left[\frac{\Gamma\left(\frac{3}{2}(1+\beta)\right)}{\Gamma\left(\frac{1}{2}(1+\beta)\right)}\right]^{\frac{1}{1+\beta}}$$

$-1 < \beta \leq 1$ AND DETERMINES DEGREE OF NONGAUSSIANITY

EXAMPLE:
$\beta = 0 \Rightarrow$ GAUSSIAN
$\beta = 1 \Rightarrow$ LAPLACIAN
$\beta \to -1 \Rightarrow \approx$ UNIFORM

FOR $\beta > 0$ TAILS ARE HEAVIER
FOR $\beta < 0$ TAILS DIE OFF MORE RAPIDLY

$$\beta = -\tfrac{1}{2} \Rightarrow p(w) \propto e^{-c_2(\beta)\left(\frac{w}{\sqrt{\sigma^2}}\right)^4}$$

SEE EXAMPLE 6.9

KNOWN DETERMINISTIC SIGNALS

EXAMPLE: DC LEVEL IN IID NONGAUSSIAN NOISE

$$\mathcal{H}_0: \; x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1: \; x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

$\uparrow$ KNOWN

$w[n]$'S ARE IID WITH KNOWN PDF $p(w[n])$

NP DECIDES $\mathcal{H}_1$ IF

$$L(\underline{x}) = \frac{p(\underline{x}; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$$

DUE TO IID

$$L(\underline{x}) = \frac{\prod_{n=0}^{N-1} p(x[n]; \mathcal{H}_1)}{\prod_{n=0}^{N-1} p(x[n]; \mathcal{H}_0)}$$

$$= \frac{\prod_{n=0}^{N-1} p(x[n]-A)}{\prod_{n=0}^{N-1} p(x[n])}$$

DECIDE $\mathcal{H}_1$ IF

$$\ln L(\underline{x}) = \sum_{n=0}^{N-1} \ln \frac{p(x[n]-A)}{p(x[n])} > \ln \gamma = \gamma'$$

LET $g(x) = \ln \frac{p(x-A)}{p(x)}$

DECIDE $\mathcal{H}_1$ IF

$$\sum_{n=0}^{N-1} g(x[n]) > \gamma'$$

CLEARLY, $g(\cdot)$ DEPENDS ON NOISE PDF.

1) GAUSSIAN

$$g(x) = \ln \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-A)^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}}$$

$$= -\frac{1}{2\sigma^2}\left[(x-A)^2 - x^2\right) = -\frac{1}{2\sigma^2}(-2Ax + A^2)$$

$$= \frac{A}{\sigma^2} x - \frac{A^2}{2\sigma^2}$$

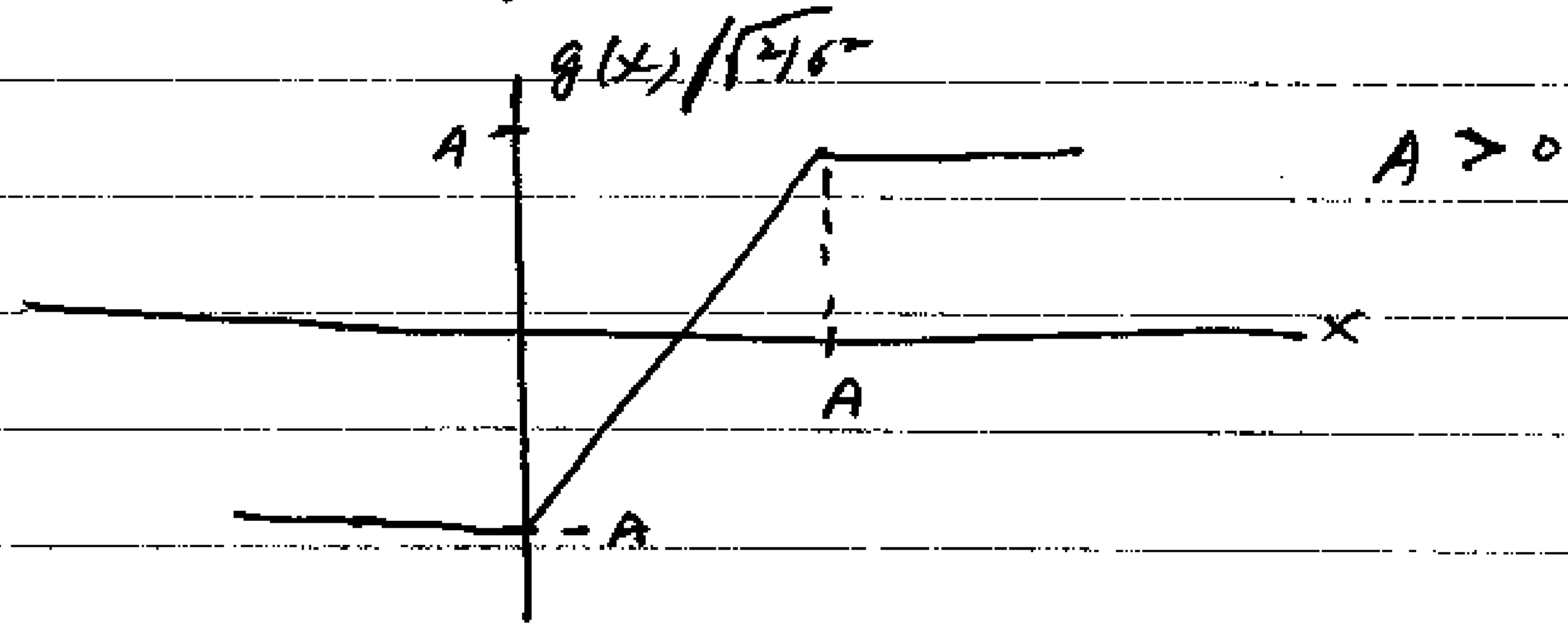
$\Rightarrow g(\cdot)$ IS LINEAR (AFFINE) IN x

STATISTIC BECOMES SAMPLE MEAN

2) LAPLACIAN

$$g(x) = \ln \frac{\frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|x-A|}}{\frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}|x|}}$$

$$= \sqrt{\frac{2}{\sigma^2}} \left(|x| - |x-A| \right)$$



$g(\cdot)$ IS NOW NONLINEAR

A MORE INTUITIVE DETECTOR RESULTS FROM LETTING $y[n] = x[n] - A/2$

$$\left.\begin{aligned} E(y[n]; \mathcal{H}_0) &= -A/2 \\ E(y[n]; \mathcal{H}_1) &= A/2 \end{aligned}\right\} \text{SYMMETRIZED VERSION}$$

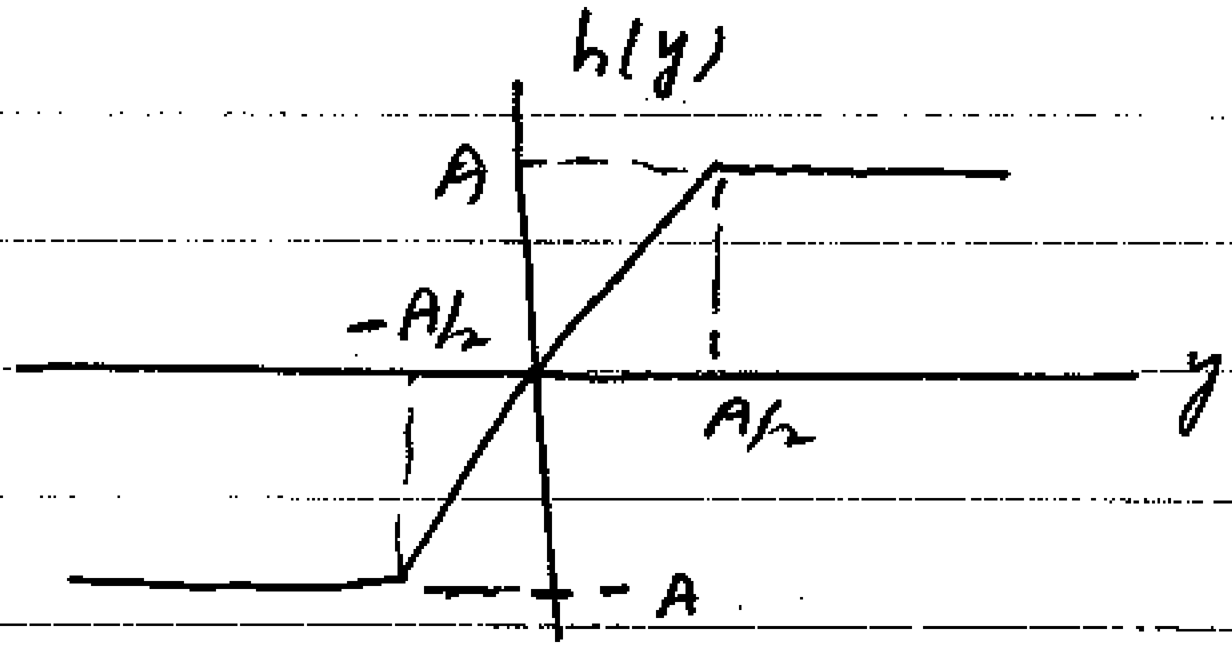
$$\sum g(x[n]) > \gamma'$$

$$\Rightarrow \sum_{n=0}^{N-1} g(y[n] + A/2) > \gamma'$$

OR $$\sum_{n=0}^{N-1} h(y[n]) > \gamma'$$

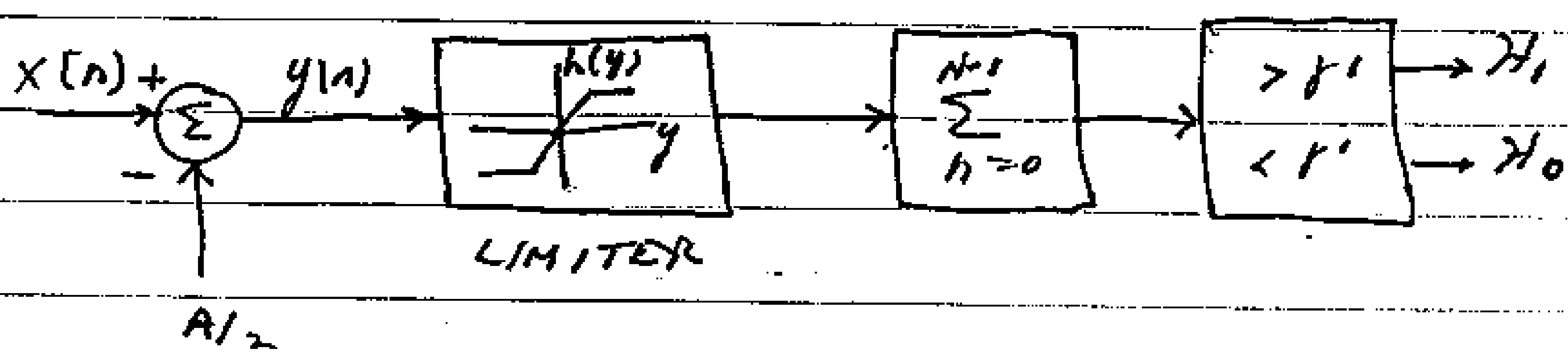
WHERE $h(y) = g(y + A/2)$

$$= \ln \frac{p(y - A/2)}{p(y + A/2)}$$



LAPLACIAN PDF

DETECTOR FIRST SUBTRACTS A/2 FROM DATA, THEN CLIPS IT, AND THEN SUMS



CLIPPER ELIMINATES NOISE SPIKES.

NOTE ALSO THAT CLIPPER IS MEMORYLESS (INDEPENDENCE ASSUMPTION) NONLINEAR TRANSFORMATION.

IN GENERAL FOR A SIGNAL $AS(n)$ WE DECIDE $\mathcal{H}_1$ IF

$$\sum_{n=0}^{N-1} g_n(x(n)) > \gamma'$$

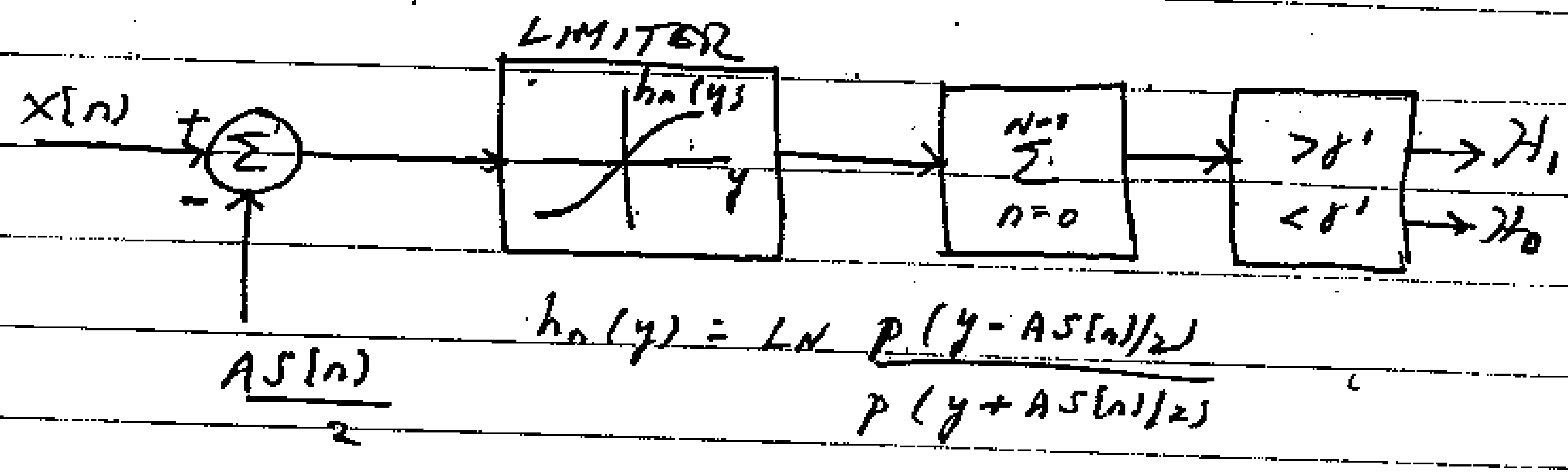
WHERE $g_n(x) = \ln \frac{p(x - As(n))}{p(x)}$

LET $y(n) = x(n) - As(n)/2$

DECIDE $\mathcal{H}_1$ IF

$$\sum_{n=0}^{N-1} h_n(y(n)) > \gamma'$$

WHERE $h_n(y) = \ln \frac{p(y - As(n)/2)}{p(y + As(n)/2)}$



$$h_n(y) = \ln \frac{p(y - AS[n]/2)}{p(y + AS[n]/2)}$$

NOTE: IF $p(w)$ IS EVEN, $h_n(y)$ WILL BE ODD $(h_n(-y) = -h_n(y))$.

DETECTION PERFORMANCE

DIFFICULT DUE TO NONLINEARITY. USE APPROXIMATION, VALID FOR WEAK SIGNALS ($A \to 0$ AND $N \to \infty$).

ASSUME SIGNAL IS $AS[n]$, WHERE $S[n]$ IS KNOWN AND $A > 0$. WHAT IS NP DETECTOR AND ITS PERFORMANCE AS $A \to 0$?

DECIDE $\mathcal{H}_1$ IF $\sum_{n=0}^{N-1} g_n(x[n]) > \gamma'$

$$g_n(x) = \ln \frac{p(x - As[n])}{p(x)}$$

AS $A \to 0$, NONLINEARITY CAN BE LINEARIZED ABOUT $A=0$

$$g_n(x) \simeq g_n(x)\Big|_{A=0} + \frac{\partial g_n(x)}{\partial A}\Big|_{A=0} A$$

$$= 0 + \frac{\frac{dp(x - As[n])}{dA}}{p(x - As[n])}\Bigg|_{A=0} A$$

$$= \frac{\frac{dp(w)}{dw}(-s[n])}{p(w)}\Bigg|_{w = x - As[n],\ A=0} A$$

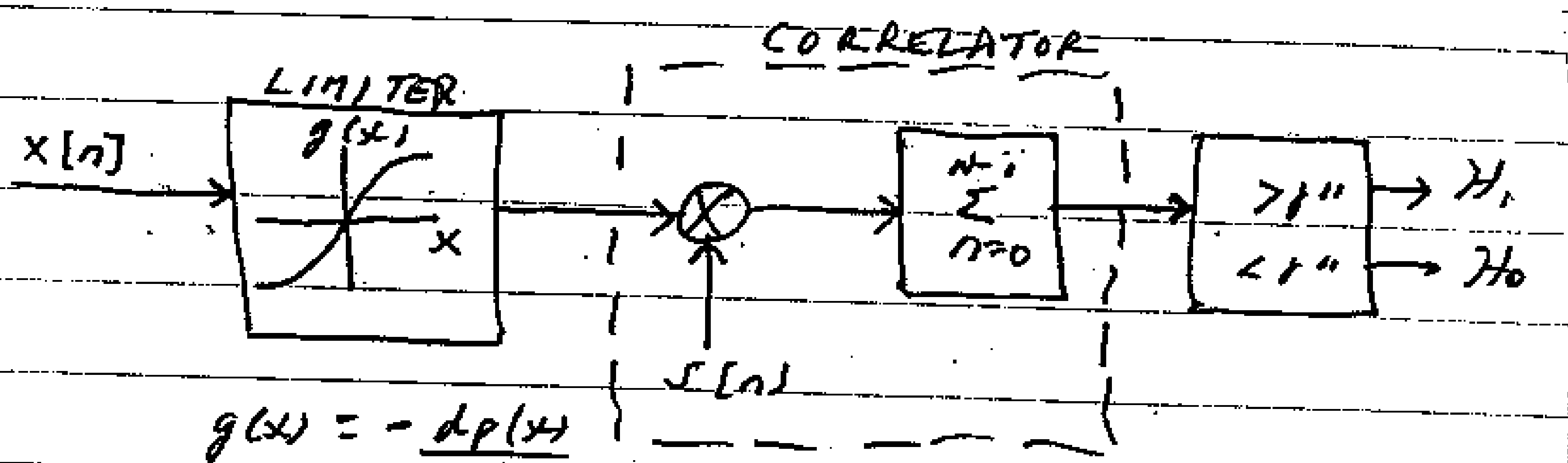
$$= -\frac{\frac{dp(x)}{dx}}{p(x)} s[n] A$$

DECIDE $\mathcal{H}_1$ IF

$$\sum_{n=0}^{N-1} g_n(x[n]) \approx \sum_{n=0}^{N-1} - \frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])} s[n] A > \gamma'$$

OR SINCE $A > 0$

$$T(\underline{x}) = \sum_{n=0}^{N-1} - \frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])} s[n] > \gamma''$$



$$g(x) = \frac{-\frac{dp(x)}{dx}}{p(x)}$$

WHAT DO WE GET FOR GAUSSIAN NOISE ?

NONLINEARITY IS NOW

$$g(x) = \frac{-\frac{dp(x)}{dx}}{p(x)} = -\frac{d \ln p(x)}{dx}$$

THIS DETECTOR IS THE SMALL SIGNAL NP DETECTOR $\Rightarrow$ CALLED LOCALLY OPTIMUM DETECTOR. IF A IS LARGE, WILL NOT BE OPTIMUM.

CAN NOW FIND PERFORMANCE (ASYMPTOTIC NP DETECTOR AS $A \to 0$, $N \to \infty$)

SEE APPENDIX 10A ($T(\underline{x})$ STILL NONLINEAR IN DATA BUT CAN USE CENTRAL LIMIT THEOREM SINCE

$$\frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])}$$ IS IID UNDER $\mathcal{H}_0$ $x[n] = w[n]$

$\approx$ IID UNDER $\mathcal{H}_1$ $x[n] = As[n] + w[n]$)

RESULT:

$$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{-\frac{dp(x[n])}{dx[n]}}{p(x[n])} s[n] = \sum_{n=0}^{N-1} g(x[n])s[n]$$

$$\overset{a}{\sim} \mathcal{N}\left(0, \; i(A) \sum_{n=0}^{N-1} s^2[n]\right) \quad \mathcal{H}_0$$

$$\mathcal{N}\left(A\,i(A) \sum_{n=0}^{N-1} s^2[n], \; i(A) \sum_{n=0}^{N-1} s^2[n]\right) \quad \mathcal{H}_1$$

WHERE $i(A) = \displaystyle\int_{-\infty}^{\infty} \frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)}\, dw$

$i(A)$ = FISHER INFORMATION FOR A BASED ON SINGLE SAMPLE.

THIS JUST MEAN-SHIFTED GAUSS-GAUSS PROBLEM

$$\Rightarrow P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

WHERE

$$d^2 = \frac{\left(E(T;\mathcal{H}_1) - E(T;\mathcal{H}_0)\right)^2}{VAR(T;\mathcal{H}_0)}$$

$$= A^2\, i(A) \sum_{n=0}^{N-1} s^2(n)$$

EXAMPLE: GAUSSIAN PDF, IID SAMPLES

DECIDE $\mathcal{H}_1$ IF

$$T(\underline{x}) = \sum_{n=0}^{N-1} g(x(n))\, s(n) > \gamma'$$

$$g(x) = \frac{-\frac{dp(x)}{dx}}{p(x)} = -\frac{d\ln p(x)}{dx}$$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}$$

$$g(x) = -\frac{d}{dx}\left(-\frac{1}{2\sigma^2}x^2\right)$$

$$= x/\sigma^2$$

$$T(\underline{x}) = 1/\sigma^2 \sum_{n=0}^{N-1} x[n]s[n]$$

USUAL CORRELATOR

(FOR THIS CASE NP $\equiv$ WEAK SIGNAL NP)

$$d^2 = A^2 i(A) \sum_{n=0}^{N-1} s^2[n]$$

$$i(A) = \int \frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)} dw$$

$$= \int \left(\frac{d \ln p(w)}{dw}\right)^2 p(w)\, dw$$

$$= E\left[\left(\frac{d \ln p(w)}{dw}\right)^2\right]$$

$$= E[g^2(w)]$$

$$= E(w^2/\sigma^4) = 1/\sigma^2$$

$$d^2 = \frac{A^2 \sum s^2(n)}{\sigma^2} = \mathcal{E}/\sigma^2$$

AS EXPECTED

ASIDE: $i(A)$ CALLED INTRINSIC ACCURACY OF PDF, IS THE FISHER INFORMATION FOR THE PARAMETER μ IF $p(x;\mu) = p_w(x-\mu)$

GAUSSIAN PDF HAS SMALLEST $i(A)$ FOR ALL PDFS WITH SAME VARIANCE $\Rightarrow$ GAUSSIAN PDF IS WORST CASE (SMALLEST d^2)

EXAMPLE: NONGAUSSIAN NOISE

CONSIDER LAPLACIAN NOISE AND A DC LEVEL $A > 0$

DECIDE $\mathcal{H}_1$ IF

$$T'(\underline{x}) = \sum_{n=0}^{N-1} \frac{-\frac{dp(x[n])}{dx[n]}}{p(x[n])} A > \gamma'$$

OR

$$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{-\frac{dp(x[n])}{dx[n]}}{p(x[n])} > \gamma''$$

BUT

$$p(w) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}\,|w|}$$

$$g(x) = \frac{-dp(x)/dx}{p(x)} = -\frac{d\ln p(x)}{dx}$$

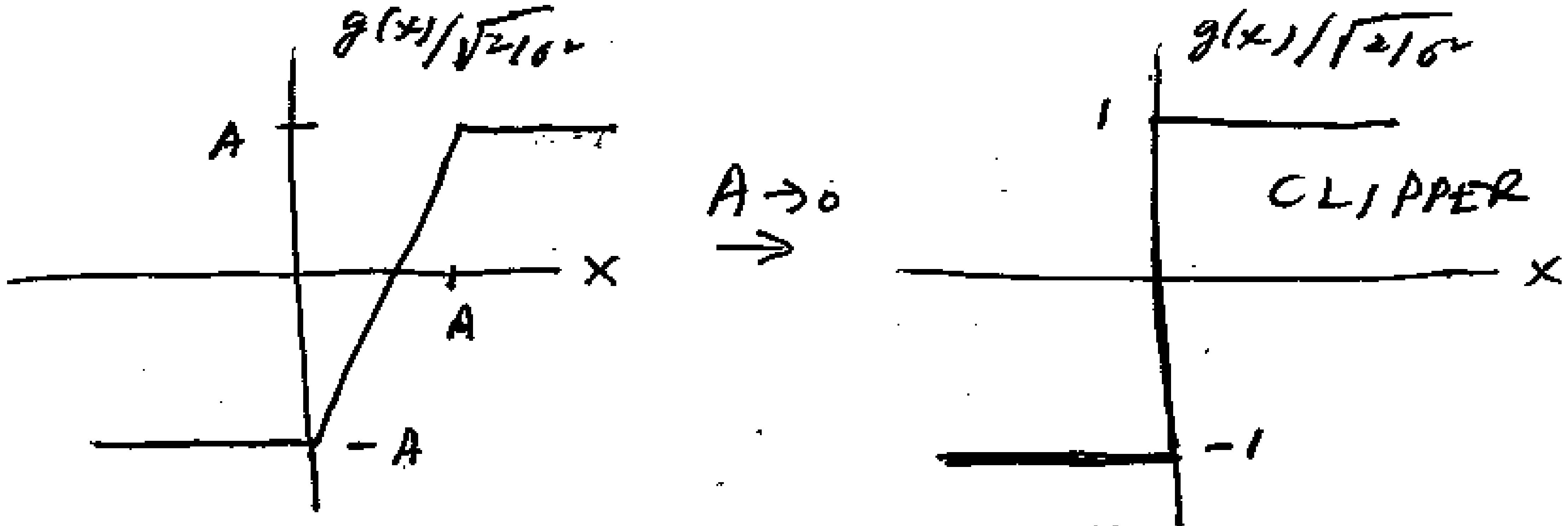
$$= -\frac{d}{dx}\left(-\sqrt{2/\sigma^2}\,|x|\right)$$

$$= \sqrt{2/\sigma^2}\,\frac{d\,|x|}{dx} = \sqrt{2/\sigma^2}\,\mathrm{SGN}(x)$$

$$T(\underline{x}) = \sqrt{2/\sigma^2} \sum_{n=0}^{N-1} \mathrm{SGN}(x[n])$$

WEAK SIGNAL DETECTOR FOR DC LEVEL ($A>0$) IN IID LAPLACIAN NOISE: ADD SAMPLE SIGNS.

CALLED A SIGN DETECTOR.



A KNOWN - NP

KNOWN OR A UNKNOWN, $A>0$ WEAK SIGNAL NP

PERFORMANCE (AS $A \to 0$) IS

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

$$\begin{aligned} d^2 &= A^2 i(A) \sum s^2(n) \\ &= A^2 i(A) N \end{aligned}$$

TO FIND $i(A)$:

$$i(A) = E(g^2(w))$$

$$= E\left(\left(\sqrt{2}/\sigma \, \mathrm{SGN}(w)\right)^2 \right)$$

$$= 2/\sigma^2 \, E(1) = 2/\sigma^2$$

= TWICE THAT FOR GAUSSIAN NOISE

$$d^2_{LAP} = \frac{2NA^2}{\sigma^2} = 2\, d^2_{GAUSSIAN}$$

DETERMINISTIC SIGNALS WITH UNKNOWN PARAMETERS

RAO TEST IS EASIER TO IMPLEMENT THAN GLRT.

CONSIDER

$$\mathcal{H}_0 : \quad x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$

$$\mathcal{H}_1 : \quad x[n] = A\, s[n] + w[n] \qquad \text{"}$$

A: UNKNOWN, $-\infty < A < \infty$

$w[n]$: IID NONGAUSSIAN $p(w[n])$