$$\underset{A}{1}\cos(2\pi f_0 n + \underset{\phi}{0}) = \underset{A}{-1}\cos(2\pi f_0 n + \underset{\phi}{\pi})$$

WILL NEED MLE OF A, ϕ. CAN BE SHOWN TO BE (FOR LARGE N AND f_0 NOT NEAR 0 OR 1/2)

$$\hat{A} = \sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}$$
$$\hat{\phi} = \arctan\left(-\frac{\hat{\alpha}_2}{\hat{\alpha}_1}\right)$$

$$\hat{\alpha}_1 = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\cos 2\pi f_0 n$$

$$\hat{\alpha}_2 = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\sin 2\pi f_0 n$$

OR $$\hat{A}e^{j\hat{\phi}} = \frac{2}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi f_0 n}$$
$$= \hat{\alpha}_1 - j\hat{\alpha}_2$$

GLRT DECIDES $\mathcal{H}_1$ IF

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \hat{\phi}, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$$

$$L_G(\underline{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}[x[n] - \hat{A}\cos(2\pi f_0 n + \hat{\phi})]^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]}}$$

$$\ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}\Big[\sum_n -2x[n]\hat{A}\cos(2\pi f_0 n + \hat{\phi}) + \sum_n \hat{A}^2\cos^2(2\pi f_0 n + \hat{\phi})\Big]$$

BUT $\displaystyle\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n + \hat{\phi}) \approx N/2$

$$\sum_{n=0}^{N-1} x[n]\hat{A}\cos(2\pi f_0 n + \hat{\phi}) =$$

$$\overbrace{\sum_n x[n]\cos 2\pi f_0 n}^{N/2\,\hat{\alpha}_1}\;\overbrace{\hat{A}\cos\hat{\phi}}^{\hat{\alpha}_1}$$

$$-\underbrace{\sum_n x[n]\sin 2\pi f_0 n}_{\frac{N}{2}\hat{\alpha}_2}\;\underbrace{\hat{A}\sin\hat{\phi}}_{-\hat{\alpha}_2}$$

$$= N/2(\hat{\alpha}_1^2 + \hat{\alpha}_2^2)$$

$$\Rightarrow \ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}\Big[-2(N/2)(\hat{\alpha}_1^2 + \hat{\alpha}_2^2) + N/2\,\underbrace{\hat{A}^2}_{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}\Big]$$

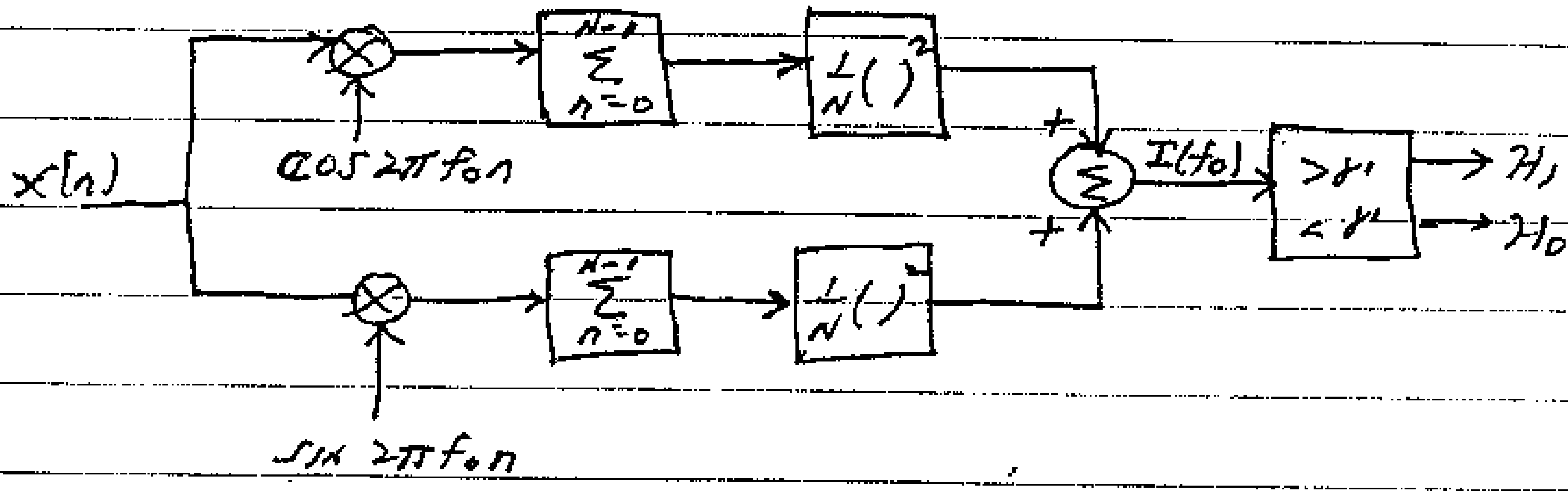
$$= -\frac{1}{2\sigma^2}\Big[-N/2\,(\hat{\alpha}_1^2 + \hat{\alpha}_2^2)\Big]$$

$$= \frac{N}{4\sigma^2}(\hat{\alpha}_1^2 + \hat{\alpha}_2^2) = \frac{N}{4\sigma^2}\hat{A}^2$$

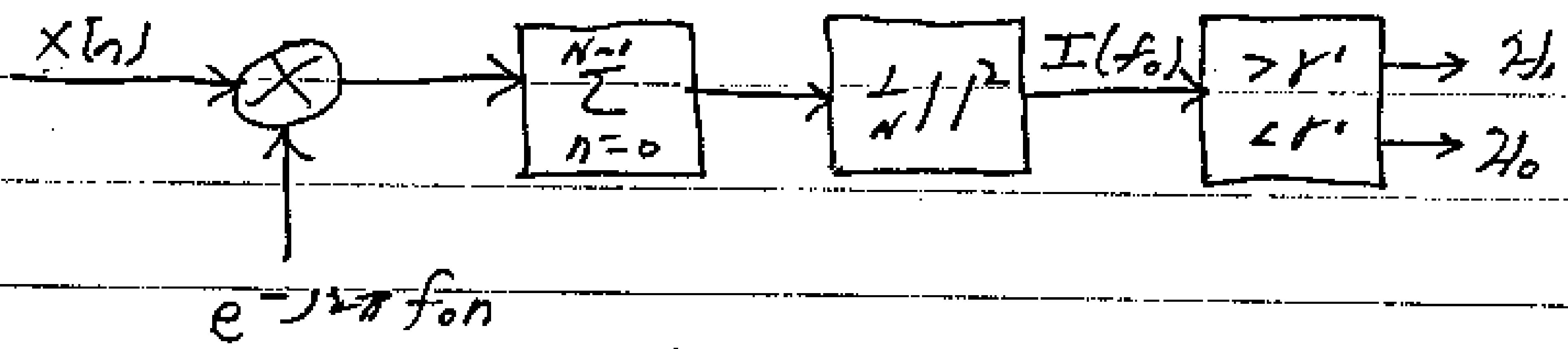
$$= \frac{N}{4\sigma^2}\left|\frac{2}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi f_0 n}\right|^2$$

$$= \frac{1}{\sigma^2} I(f_0)$$

↑ PERIODOGRAM!



QUADRATURE MATCHED FILTER



PERIODOGRAM

b) UNKNOWN AMPLITUDE AND PHASE

PERFORMANCE WILL BE DIFFERENT THAN FOR RAYLEIGH FADING

HERE: A, ϕ DETERMINISTIC AND UNKNOWN

FADING: $A \sim$ RAYLEIGH

$\phi \sim$ UNIFORM

TO FIND PERFORMANCE:

$$I(f_0) = \xi_1^2 + \xi_2^2$$

$$\xi_1 = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \cos 2\pi f_0 n$$

$$\xi_2 = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \sin 2\pi f_0 n$$

OR $\underline{\xi} = \underline{H}^T \underline{x}$ $\qquad \underline{H} = \begin{bmatrix} \frac{1}{\sqrt{N}} & 0 \\ \frac{1}{\sqrt{N}} \cos 2\pi f_0 & \frac{1}{\sqrt{N}} \sin 2\pi f_0 \\ \vdots & \vdots \end{bmatrix}$

$\underline{\xi} \sim$ GAUSSIAN SINCE $\underline{x}$ IS GAUSSIAN $\qquad \underline{H}^T\underline{H} \approx \frac{1}{2}\underline{I}$

UNDER $\mathcal{H}_0$:

$$\underline{x} \sim N(\underline{0}, \sigma^2 \underline{I})$$

$$\Rightarrow \underline{\xi} \sim N(\underline{0}, \underbrace{\underline{H}^T \sigma^2 \underline{I}\, \underline{H}}_{\frac{\sigma^2}{2}\underline{I}})$$

UNDER $\mathcal{H}_1$:

$$\underline{z} \sim N\left(\underline{H}^T E(\underline{x}),\ \frac{\sigma^2}{2} \underline{I}\right)$$

$$E(\underline{x}) = \begin{bmatrix} A\cos\phi \\ A\cos(2\pi f_0 + \phi) \\ \vdots \\ A\cos(2\pi f_0 (N-1) + \phi) \end{bmatrix}$$

$$\Rightarrow E(z_1) = \frac{A}{\sqrt{N}} \sum_{n=0}^{N-1} \cos 2\pi f_0 n \cos(2\pi f_0 n + \phi)$$

$$= \frac{A}{2\sqrt{N}} \sum_n \left[\cos\phi + \cos(4\pi f_0 n + \phi)\right]$$

$$\simeq \frac{\sqrt{N} A}{2} \cos\phi$$

SIMILARLY $E(z_2) \simeq -\frac{\sqrt{N} A}{2} \sin\phi$

$$\underline{z} \sim N\left(\begin{bmatrix} \frac{\sqrt{N}A}{2}\cos\phi \\ -\frac{\sqrt{N}A}{2}\sin\phi \end{bmatrix},\ \frac{\sigma^2}{2}\underline{I}\right)$$

NOTE z_1, z_2 ARE INDEPENDENT UNDER $\mathcal{H}_0$ AND $\mathcal{H}_1$.

DECIDE $\mathcal{H}_1$ IF $I(f_0) = z_1^2 + z_2^2 > \gamma'$

UNDER $\mathcal{H}_0$ $$\frac{z_1^2 + z_2^2}{\sigma^2/2} = \underbrace{\left(\frac{z_1}{\sqrt{\sigma^2/2}}\right)^2}_{N(0,1)} + \underbrace{\left(\frac{z_2}{\sqrt{\sigma^2/2}}\right)^2}_{N(0,1)}$$

$$\sim \chi_2^2$$

$$\begin{aligned}
P_{FA} &= P_r\{ I(f_0) > \gamma' ; \mathcal{H}_0 \} \\
&= P_r\Big\{ \underbrace{\frac{I(f_0)}{\sigma^2/2}}_{\chi_2^2} > \frac{\gamma'}{\sigma^2/2} ; \mathcal{H}_0 \Big\} \\
&= \int_{\frac{\gamma'}{\sigma^2/2}}^{\infty} \frac{1}{2} e^{-\frac{1}{2}x}\, dx \\
&= e^{-\gamma'/\sigma^2}
\end{aligned}$$

UNDER $\mathcal{H}_1$

$$\frac{\xi_1^2 + \xi_2^2}{\sigma^2/2} = \Big(\underbrace{\frac{\xi_1}{\sqrt{\sigma^2/2}}}_{N\left(\frac{\mu_1}{\sqrt{\sigma^2/2}},\, 1\right)}\Big)^2 + \Big(\underbrace{\frac{\xi_2}{\sqrt{\sigma^2/2}}}_{N\left(\frac{\mu_2}{\sqrt{\sigma^2/2}},\, 1\right)}\Big)^2$$

$$\sim \chi_2'^2(\lambda)$$

WHERE $\lambda = \left(\frac{\mu_1}{\sqrt{\sigma^2/2}}\right)^2 + \left(\frac{\mu_2}{\sqrt{\sigma^2/2}}\right)^2$

$$\begin{aligned}
&= \frac{\frac{NA^2}{4}\cos^2\phi}{\sigma^2/2} + \frac{\frac{NA^2}{4}\sin^2\phi}{\sigma^2/2} \\
&= \frac{NA^2}{2\sigma^2}
\end{aligned}$$

$$\Rightarrow P_D = \Pr\{ I(f_0) > \gamma' ; \mathcal{H}_1 \}$$

$$= \Pr\{ \underbrace{\frac{I(f_0)}{\sigma^2/2}}_{\chi_2'^2(\lambda)} > \frac{\gamma'}{\sigma^2/2} ; \mathcal{H}_1 \}$$

$$= Q_{\chi_2'^2(\lambda)} \left(2\gamma'/\sigma^2 \right)$$

BUT FROM CHAPTER 2

$$P_D = \sum_{k=0}^{\infty} \frac{e^{-\lambda/2} (\lambda/2)^k}{k!} Q_{\chi^2_{2+2k}} \left(\frac{2\gamma'}{\sigma^2} \right)$$

OR USE CHIPR2.M

$$P = \text{CHIPR2}\left(2, \lambda, \frac{2\gamma'}{\sigma^2}, \epsilon \right)$$

λ: $\frac{NA^2}{2\sigma^2}$ $\qquad$ ϵ: MAX ERROR ALLOWABLE

SEE FIG. 7.6 b FOR P_D CURVES. (COMPARE TO FIG. 7.6 a FOR ϕ KNOWN).

3) A, ϕ, f_0 UNKNOWN

GLRT DECIDES $\mathcal{H}_1$ IF

$$\frac{p(\underline{x};\hat{A},\hat{\phi},\hat{f}_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

RECALL THAT FOR θ UNKNOWN
MLE OF θ IS VALUE THAT MAXIMIZES
$p(\underline{x};\theta)$ OR $p(\underline{x};\hat{\theta}) > p(\underline{x};\theta)$
FOR ALL $\theta \neq \hat{\theta}$. THUS

$$p(\underline{x};\hat{\theta}) = \max_{\theta} p(\underline{x};\theta)$$

$$p(\underline{x};\hat{A},\hat{\phi},\hat{f}_0,\mathcal{H}_1) = \max_{f_0} p(\underline{x};\hat{A},\hat{\phi},f_0,\mathcal{H}_1)$$

GLRT DECIDES $\mathcal{H}_1$ IF

$$\frac{\max_{f_0} p(\underline{x};\hat{A},\hat{\phi},f_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

$$\max_{f_0}\left[\frac{p(\underline{x};\hat{A},\hat{\phi},f_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}\right] > \gamma$$

$$\downarrow$$

$$\ln \max_{f_0} [\quad] > \ln \gamma$$

BUT $\ln \max_{x} g(x) = \max_{x} \ln g(x)$

SINCE LN IS MONOTONICALLY INCREASING

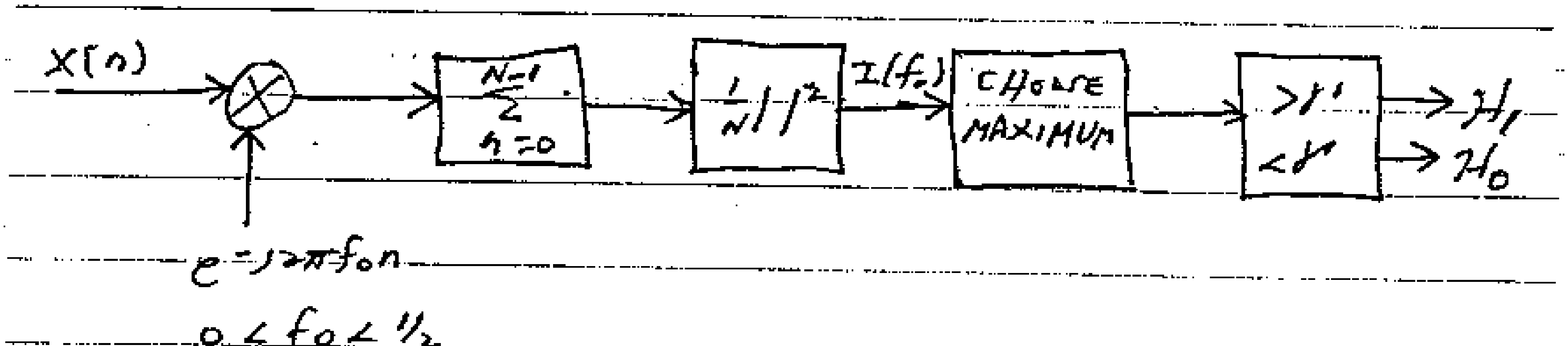
$$\Rightarrow \max_{f_0} \underbrace{\ln \frac{p(\underline{x};\hat{A},\hat{\phi},f_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}}_{I(f_0)/\sigma^2} > \ln \gamma$$

A, φ UNKNOWN CASE

WE DECIDE $\mathcal{H}_1$ IF

$$\max_{f_0} I(f_0) > \sigma^2 \ln \gamma = \gamma'$$

CONVENTIONAL NARROWBAND OR FFT DETECTOR



c) UNKNOWN AMPLITUDE, PHASE, FREQUENCY

WILL DISCUSS PERFORMANCE LATER.

4) A, ϕ, f_0, n_0 UNKNOWN

GLRT DECIDES $\mathcal{H}_1$ IF

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \hat{\phi}, \hat{f}_0, \hat{n}_0; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$$

OR

$$L_G(\underline{x}) = \max_{n_0} \frac{p(\underline{x}; \hat{A}, \hat{\phi}, \hat{f}_0, n_0; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)}$$

EXCEPT FOR MAXIMIZATION OVER n_0 WE HAVE SAME GLRT AS FOR A, ϕ, f_0 UNKNOWN. NEED ONLY MODIFY OBSERVATION INTERVAL TO $[n_0, n_0+M-1]$ (BEFORE $n_0=0$, $M=N$).

$$\Rightarrow I_{n_0}(f_0) = \frac{1}{M} \left| \sum_{n=n_0}^{n_0+M-1} x[n]\, e^{-j2\pi f_0 n} \right|^2$$

$$\ln \frac{p(\underline{x}; \hat{A}, \hat{\phi}, \hat{f}_0, n_0, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} = \frac{I_{n_0}(\hat{f}_0)}{\sigma^2}$$

FROM UNKNOWN A, ϕ, f_0 CASE.

$$L_G(\underline{x}) = \max_{n_0} \frac{I_{n_0}(\hat{f}_0)}{\sigma^2}$$

$$= \max_{n_0, f_0} \frac{I_{n_0}(f_0)}{\sigma^2}$$

WHERE

$$I_{n_0}(f_0) = \frac{1}{M} \left| \sum_{n=n_0}^{n_0+M-1} x[n] e^{-j2\pi f_0 n} \right|^2$$

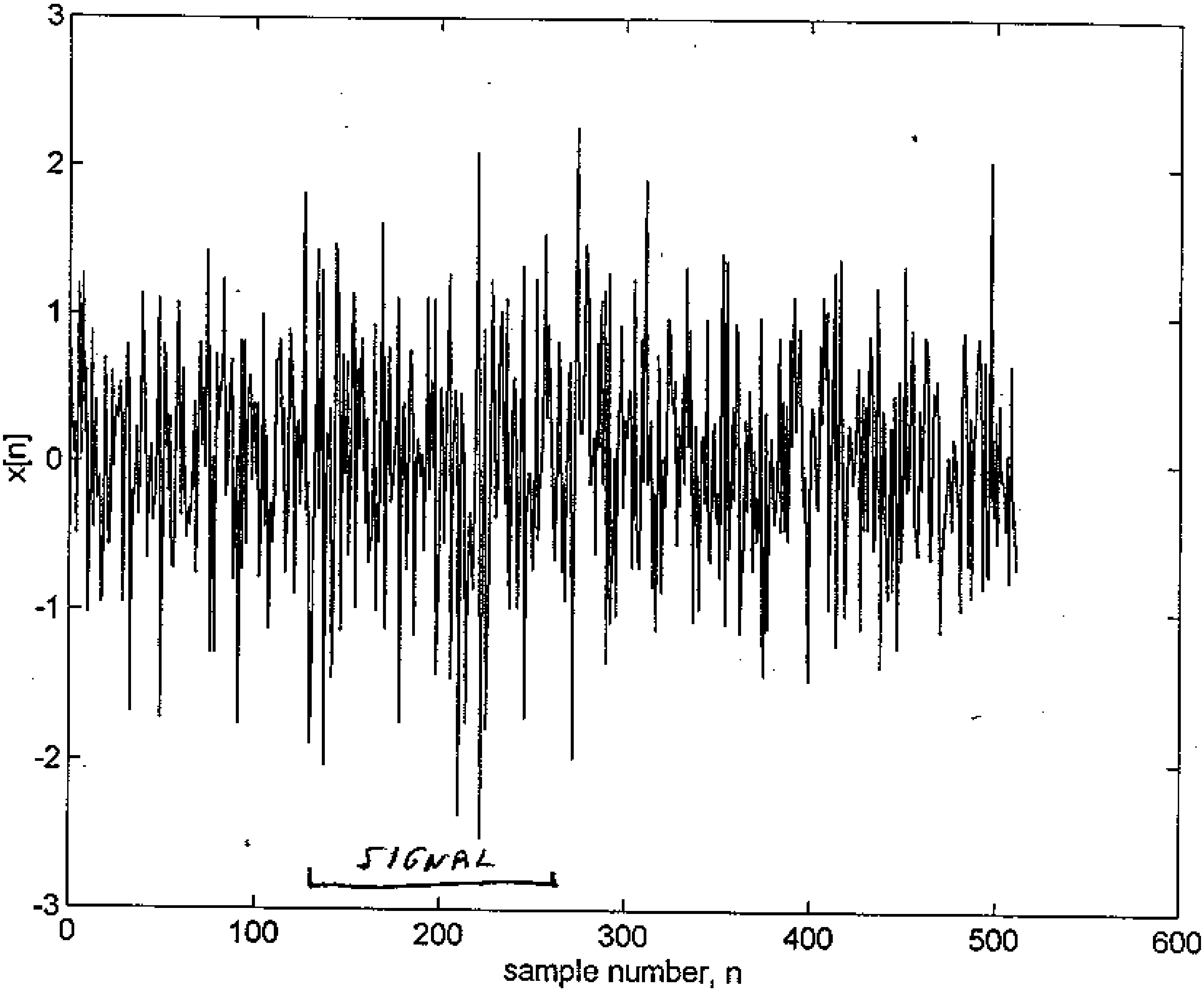
THIS IS A "RUNNING" FOURIER TRANSFORM. CALLED SHORT-TIME PERIODOGRAM OR SPECTROGRAM.

GLRT PICKS PEAK OF $I_{n_0}(f_0)$ AND COMPARES TO THRESHOLD. IF EXCEEDED $\Rightarrow$ MLE OF DELAY AND FREQUENCY

EXAMPLE: $A = 1$, $f_0 = 0.25$, $\phi = 0$, $M = 128$, $n_0 = 128$, $N = 512$, $\sigma^2 = 0.5$

$$SNR = A^2/2\sigma^2 = 1 \text{ OR } 0 \text{ dB}$$

SIGNAL AT [128, 255]



7.7a) TIME SERIES DATA

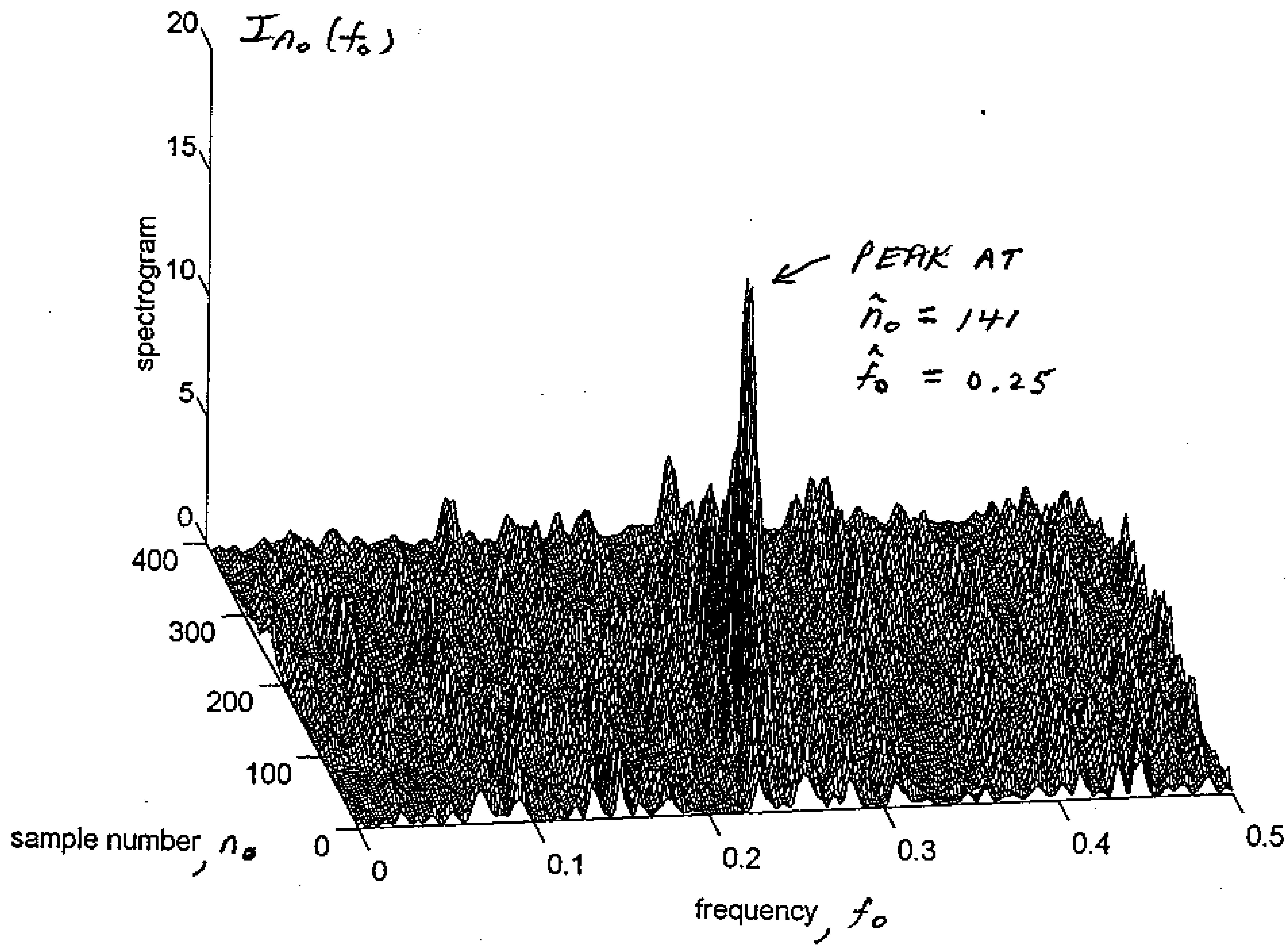
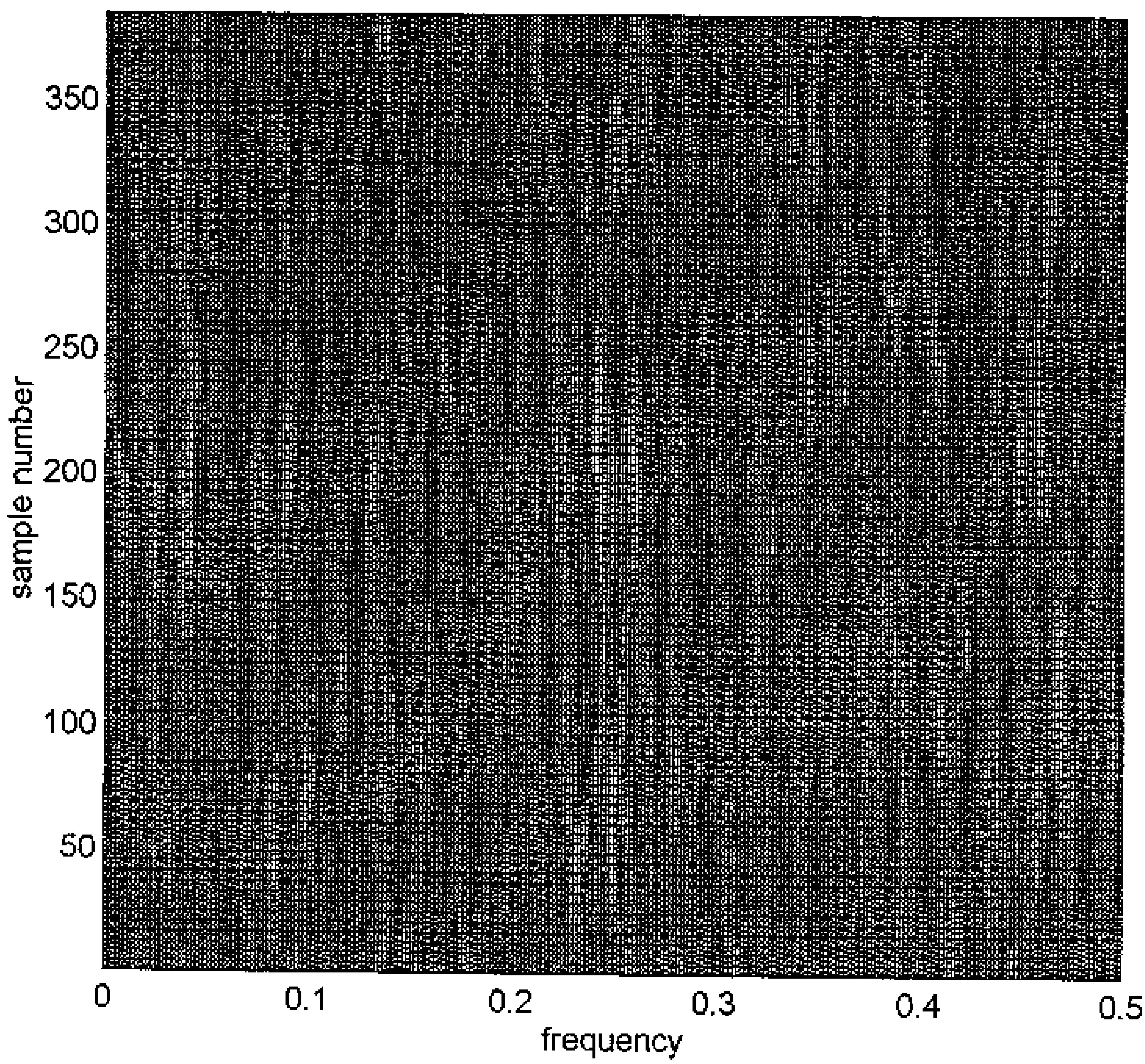


FIGURE 7.7b - SPECTROGRAM OF DATA



7.7c - GRAY SCALE SPECTROGRAM OF DATA

CLASSICAL LINEAR MODEL

RECALL MODEL $\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$

$$\underbrace{\underline{x}}_{N\times 1} = \underbrace{\underline{H}}_{N\times p}\,\underbrace{\underline{\theta}}_{p\times 1} + \underbrace{\underline{w}}_{N\times 1}$$

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

$\underline{H}$ IS KNOWN

$\underline{x}$ IS DATA VECTOR

$\underline{\theta}$ IS TO BE TESTED

IN GENERAL WE CONSIDER

$$\mathcal{H}_0 : \underline{A}\,\underline{\theta} = \underline{b}$$

$$\mathcal{H}_1 : \underbrace{\underline{A}}_{r\times p}\,\underline{\theta} \neq \underbrace{\underline{b}}_{r\times 1} \qquad r \le p$$

UNDER $\mathcal{H}_0$ $\underline{\theta}$ LIES IN SOME "SUBSPACE" OF R^p

EXAMPLE: SIGNAL WITH UNKNOWN AMPLITUDE

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, 1, \ldots, N-1$$

$$\mathcal{H}_1 : x[n] = A s[n] + w[n] \qquad \text{"}$$

OR $\mathcal{H}_0: A = 0$

$\mathcal{H}_1: A \neq 0$

$\underline{\theta} = A, \quad \underbrace{\underline{A}}_{1\times 1} = \underline{I}, \quad \underbrace{\underline{b}}_{1\times 1} = \underline{0}$

$\underline{x} = [x[0] \ldots x[N-1]]^T, \quad \underline{H} = [s[0] \ldots s[N-1]]^T$

$\underline{w} = [w[0] \ldots w[N-1]]^T$

THEOREM: GLRT FOR CLASSICAL LINEAR MODEL

IF $\underline{x} = \underline{H}\underline{\theta} + \underline{w}$, $\underline{H}$ IS KNOWN $N \times p$ MATRIX, $\underline{\theta}$ IS $p \times 1$, $\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$, TO TEST

$\mathcal{H}_0: \underline{A}\underline{\theta} = \underline{b}$

$\mathcal{H}_1: \underline{A}\underline{\theta} \neq \underline{b}$

$\underline{A}$ IS $r \times p$ $(r \leq p)$

$\underline{b}$ IS $r \times 1$

$\underline{A}\underline{\theta} = \underline{b}$ CONSISTENT ($\Rightarrow$ AT LEAST ONE SOLUTION)

GLRT DECIDES $\mathcal{H}_1$ IF

$$T(\underline{x}) = \frac{(\underline{A}\hat{\underline{\theta}}_1 - \underline{b})^T [\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T]^{-1} (\underline{A}\hat{\underline{\theta}}_1 - \underline{b})}{\sigma^2} > \gamma$$

WHERE $\hat{\theta}_1 = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$

IS MLE OF $\underline{\theta}$ UNDER $\mathcal{H}_1$

(NO RESTRICTIONS).

DETECTION PERFORMANCE IS

(EXACTLY, NOT AS $N \to \infty$)

$$P_{FA} = Q_{\chi^2_r}(\gamma')$$

$$P_D = Q_{\chi'^2_r(\lambda)}(\gamma')$$

WHERE $$\lambda = \frac{(\underline{A}\theta_1 - \underline{b})^T[\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T]\cdot(\underline{A}\theta_1 - \underline{b})}{\sigma^2}$$

SEE APPENDIX 7B FOR PROOF.

EXAMPLE: DC OFFSET COMPENSATION, SIGNAL WITH UNKNOWN A

$\mathcal{H}_0$: $x[n] = \mu + w[n]$ $\quad n = 0, 1, \ldots, N-1$

$\mathcal{H}_1$: $x[n] = A s[n] + \mu + w[n]$ $\quad$ "

A UNKNOWN

μ DC OFFSET - UNKNOWN

$w[n]$ IS WGN WITH KNOWN σ^2

UNDER $\mathcal{H}_1$ IN FORM OF LINEAR MODEL

$$\underline{x} = \underbrace{\begin{bmatrix} s(0) & 1 \\ s(1) & 1 \\ \vdots & \vdots \\ s(N-1) & 1 \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} A \\ \mu \end{bmatrix}}_{\underline{\theta}} + \underline{w}$$

DETECTION PROBLEM IS

$$\mathcal{H}_0 : A = 0 \quad , \mu$$
$$\mathcal{H}_1 : A \neq 0 \quad , \mu$$

↑ NUISANCE PARAMETER

OR $\mathcal{H}_0 : [1 \ 0] \begin{bmatrix} A \\ \mu \end{bmatrix} = 0$

$$\mathcal{H}_1 : \underbrace{[1 \ 0]}_{\underset{1 \times 2}{\underline{A}}} \underbrace{\begin{bmatrix} A \\ \mu \end{bmatrix}}_{\underset{2 \times 1}{\underline{\theta}}} \neq \underbrace{0}_{\underset{1 \times 1}{\underline{b}}} \Rightarrow r = 1, \ p = 2$$

THEOREM APPLIES :

GLRT DECIDES $\mathcal{H}_1$ IF

$$T(\underline{x}) = \frac{\hat{\underline{\theta}}_1^T \underline{A}^T [\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T]^{-1} \underline{A}\hat{\underline{\theta}}_1}{\sigma^2} > \gamma$$

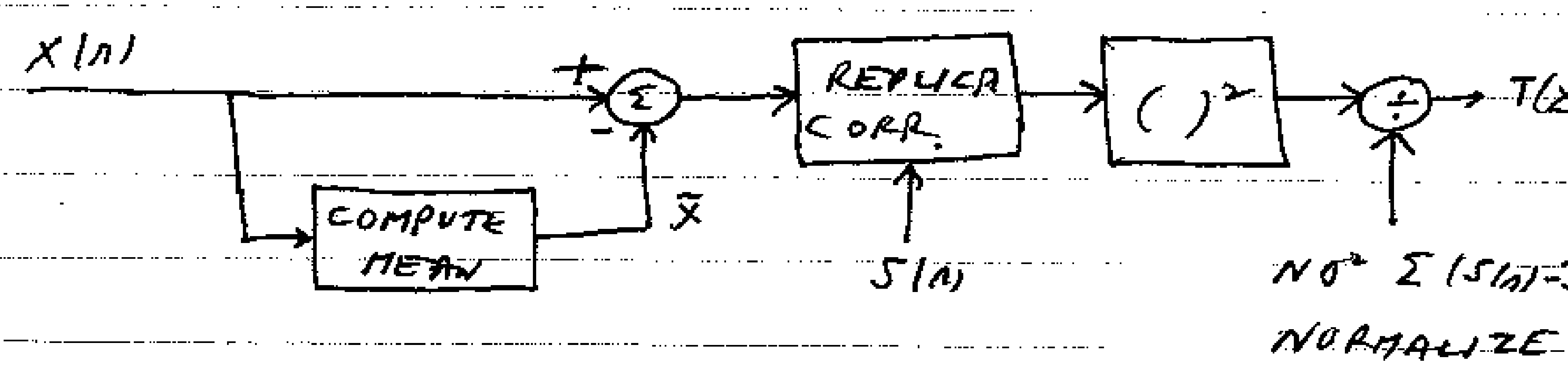
$$\hat{\underline{\theta}}_1 = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$$

$$\underline{A} = [1 \;\; 0]$$

AFTER SOME ALGEBRA (SEE BOOK)

$$T(\underline{x}) = \frac{\left(\sum_{n=0}^{N-1}(x[n]-\bar{x})s[n]\right)^2}{N\sigma^2 \sum_{n=0}^{N-1}(s[n]-\bar{s})^2}$$

WHERE $\bar{s} = \frac{1}{N}\sum_{n=0}^{N-1} s[n]$



SINCE $\Gamma = 1$

$$P_{FA} = Q_{\chi^2_1}(\gamma') \Rightarrow \text{CAN FIND THRESHOLD}$$

NOTE THAT $T(\underline{x})$ IS INDEPENDENT OF $\bar{s}$ SINCE

$$T(\underline{x}) = \frac{\left(\sum_{n=0}^{N-1}(x[n]-\bar{x})(s[n]-\bar{s})\right)^2}{N\sigma^2 \sum_{n=0}^{N-1}(s[n]-\bar{s})^2}$$

WHY?

ACTIVE SONAR / RADAR DETECTION

NOW CONSIDER A TYPICAL SONAR NARROWBAND DETECTOR (ACTIVE)

TRANSMIT $s(t) = B\cos(2\pi F_T t + \phi_T)$

1) DUE TO ATTENUATION OF MEDIUM
   RECEIVE $A\cos(2\pi F_T t + \phi_T)$ $(A \neq B)$
2) DUE TO REFLECTIONS OFF TARGET, SEA SURFACE, ETC.
   RECEIVE $A\cos(2\pi F_T t + \phi)$ $(\phi \neq \phi_T)$
3) DUE TO DOPPLER EFFECT
   RECEIVE $A\cos(2\pi F_0 t + \phi)$ $(F_0 \neq F_T)$
4) AFTER SAMPLING WE HAVE
   $$A\cos(2\pi \underbrace{F_0 \Delta}_{f_0} n + \phi)$$

A, f_0, ϕ UNKNOWN AS WELL AS ARRIVAL TIME $\Rightarrow$ GLRT IS SPECTROGRAM OR DECIDE $\mathcal{H}_1$ IF

$$\max_{n_0, f_0} \frac{1}{M} \left| \sum_{n=n_0}^{n_0+M-1} x[n]\, e^{-j2\pi f_0 n} \right|^2 > \gamma'$$

ASSUMING WGN OF VARIANCE σ^2.

CHANGE NOTATION FOR SIGNAL LENGTH TO N

TO COMPUTE SPECTROGRAM USE FFT OR EVALUATE f_0 FOR

$f_0 = k/N \quad k = 1, 2, \ldots, N/2 - 1$

(N EVEN)

(ASSUMES $f_0 \neq 0$ OR $1/2$)

DECIDE $\mathcal{H}_1$ IF

$$\max_{\substack{n_0 \\ k = 1, 2, \ldots, \frac{N}{2} - 1}} \frac{1}{N} \left| \sum_{n=n_0}^{n_0+N-1} x[n] \, e^{-j \frac{2\pi}{N} k n} \right|^2 > \gamma'$$

SOME LOSS IF $f_0 \neq k/N$ – CALLED SCALLOPING LOSS

$$\left| \sum_{n=n_0}^{n_0+N-1} A \cos(2\pi f_0 (n - n_0) + \phi) \, e^{-j \frac{2\pi}{N} k n} \right|^2$$

= MAGNITUDE-SQUARED OF FOURIER TRANSFORM OF SINUSOID EVALUATED AT $f = k/N$