

ELE 665 - DETECTION THEORY

INSTRUCTOR : PROF. STEVEN KAY

TEXT : FUNDAMENTALS OF STATISTICAL
SIGNAL PROCESSING - DETECTION
S. KAY

REFERENCES : 1) ADVANCED THEORY OF
STATISTICS VOL 2,
KENDALL & STUART
2) DETECTION OF SIGNALS
IN NOISE, McDONOUGH & WHALEN
3) DETECTION, ESTIMATION, AND
MODULATION THEORY, VOL 1,
VAN TREES

COURSE OUTLINE : SEE TABLE OF CONTENTS
STATISTICAL DECISION THEORY,
NEYMAN-PEARSON, BAYES, KNOWN
SIGNALS, RANDOM SIGNALS, GLRT
AND RAD TESTS, NONGAUSSIAN NOISE

PREREQUISITES : 1) ELE 509 (SYSTEMS WITH
RANDOM INPUTS)

- 2) ELE 661 - ESTIMATION
- 3) LINEAR ALGEBRA + MATRIX ALGEBRA

GRADING: 1) HOMEWORK ASSIGNED WEEKLY
 DUE FOLLOWING WEEK - NOT GRADED BUT REASONABLE EFFORT REQUIRED

2) DATA ANALYSIS (20%)
 FINAL WRITTEN REPORT

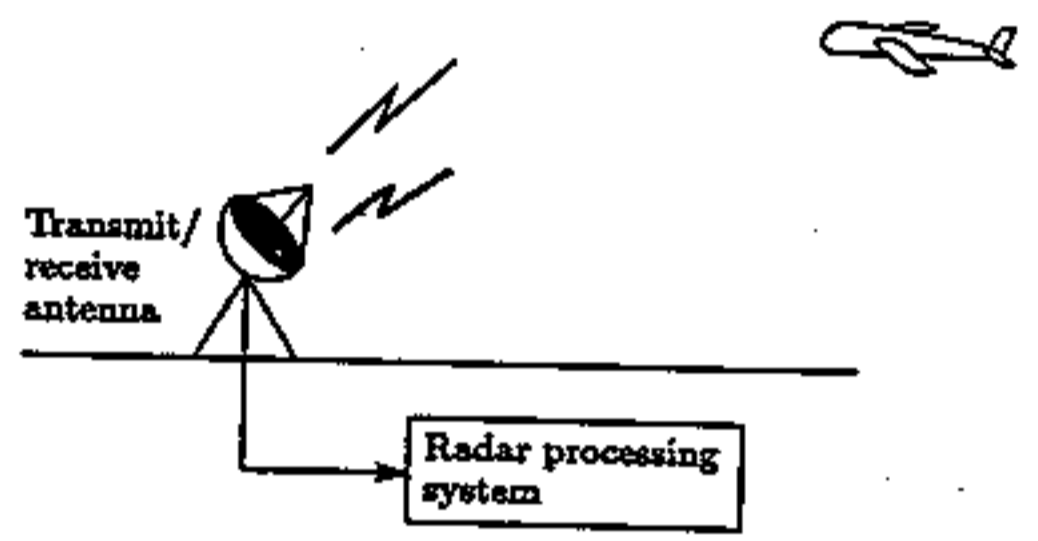
3) PROJECT (80%)
 a) LITERATURE SURVEY
 b) ANALYSIS (ORIGINAL RESEARCH)
 c) COMPUTER SIMULATION
 PROPOSAL, ORAL PRESENTATION,
 FINAL WRITTEN REPORT

OFFICE HOURS: M, W, F 9-11 874-5804
 KELLEY ANNEX A123

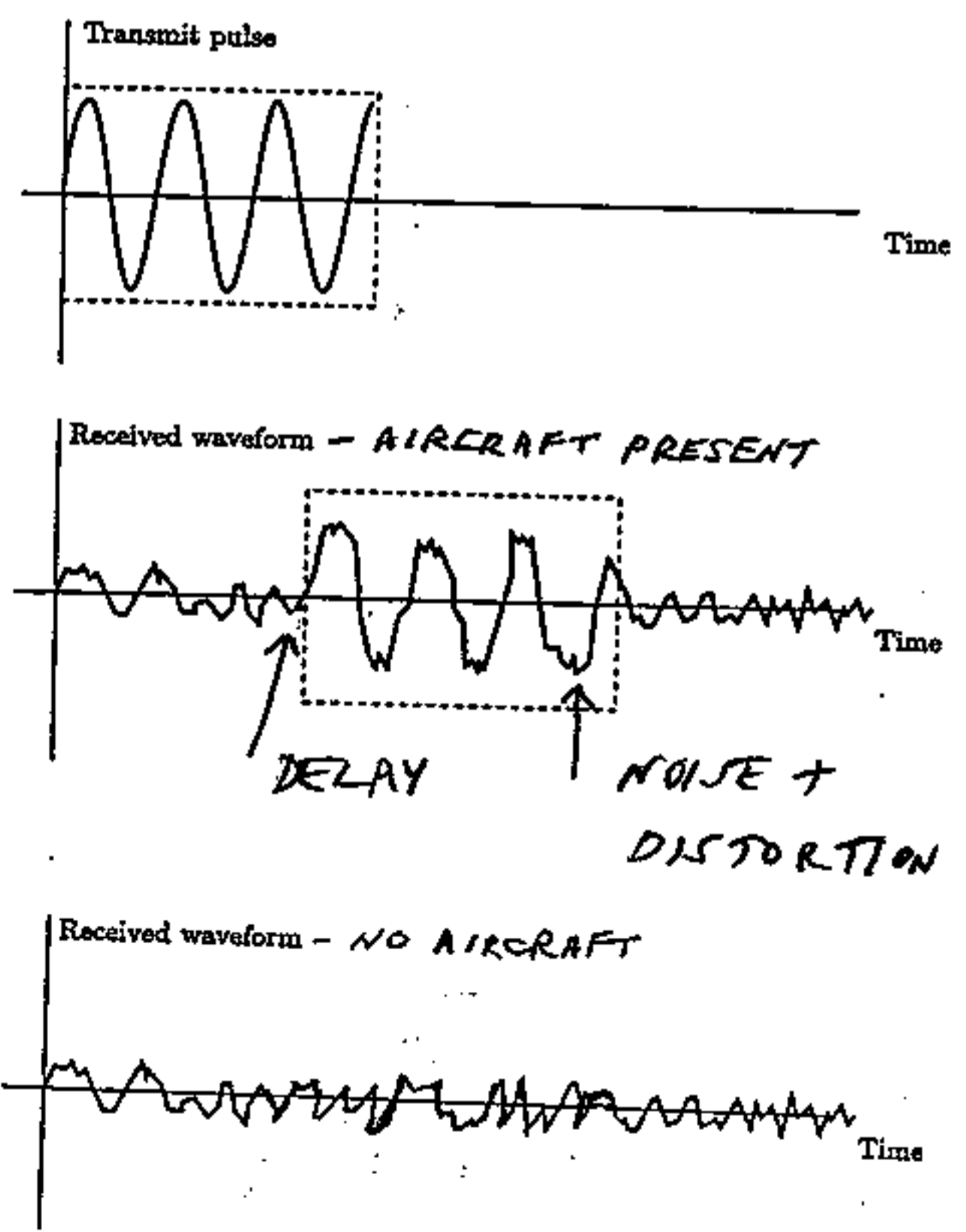
INTRODUCTION

APPLICATION AREAS

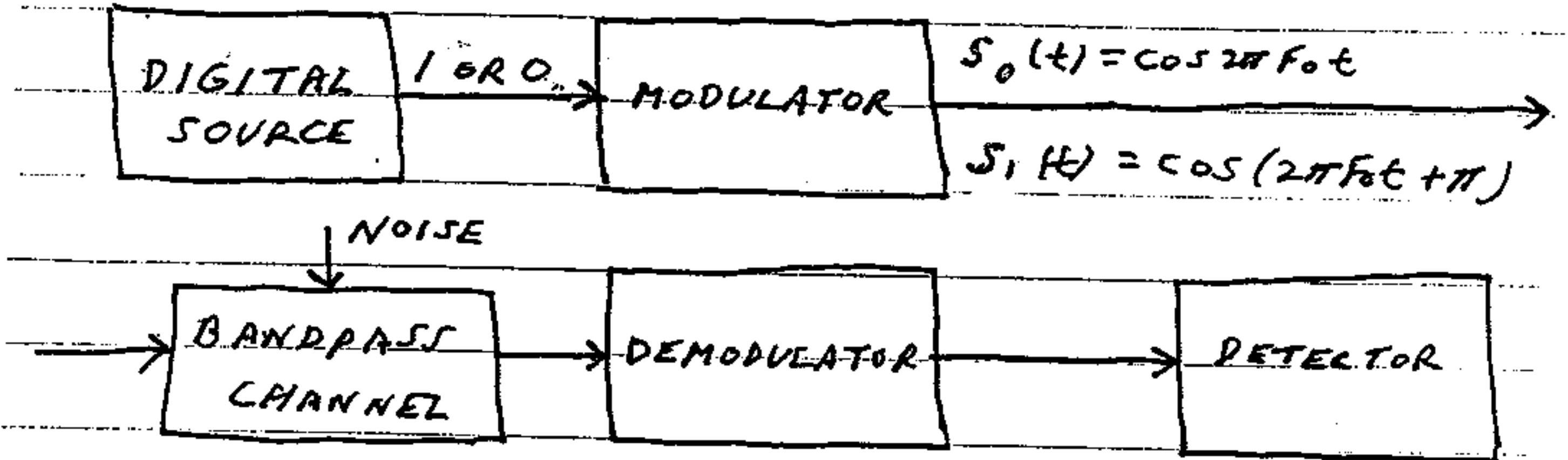
- 1) RADAR - DETERMINE IF AIRCRAFT IS APPROACHING



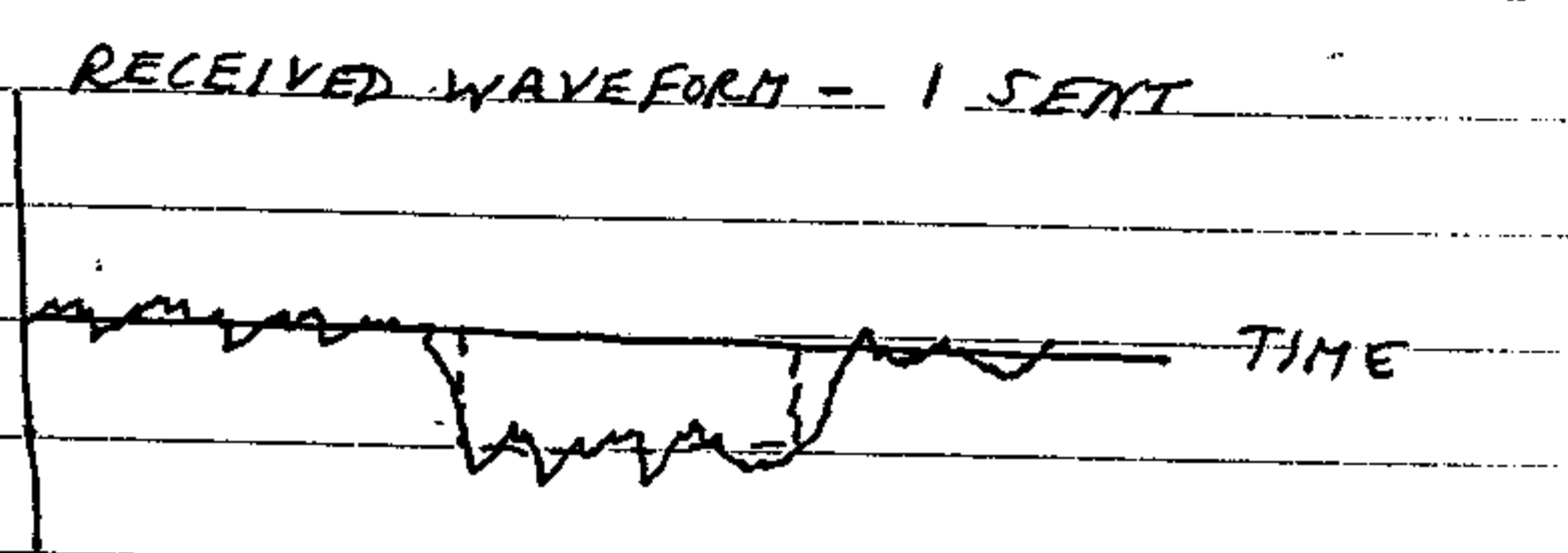
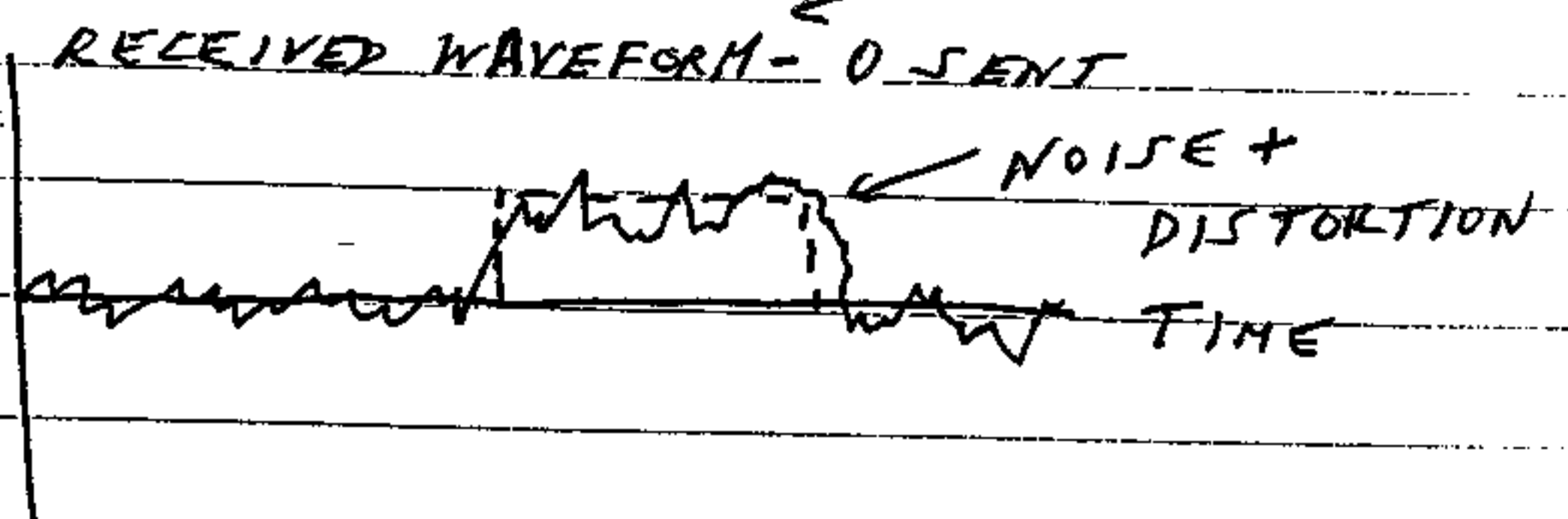
(a) Radar



2) COMMUNICATIONS - BPSK



a) BASIC SYSTEM AT DEMODULATOR OUTPUT



3) SPEECH - RECOGNIZE DIGITS,
0 → 9

WAVEFORMS DEPEND ON SPEAKER
AND CHANGE FOR A GIVEN DIGIT
SEE FIG. 1.3

4) SONAR - DETECT ENEMY SUBMARINE

5) IMAGE PROCESSING - DETECT HELICOPTER
VIA INFRARED SURVEILLANCE

6) BIOMED - DETECT CARDIAC ARRHYTHMIA

7) CONTROL - DETECT ABRUPT CHANGE
IN CONTROLLED SYSTEM

8) SEISMIC - DETECT OIL DEPOSIT
ETC.

DETECTION PROBLEM

SIMPLEST - SIGNAL PRESENT VS.
SIGNAL ABSENT (RADAR)

TWO POSSIBLE CHOICES \Rightarrow

BINARY HYPOTHESIS TESTING
PROBLEM

ANOTHER EXAMPLE IS BPSK

$$s_0(t) = \cos 2\pi F_0 t$$

VS. $s_1(t) = \cos(2\pi F_0 t + \pi)$

NOW WE HAVE A SIGNAL UNDER
EITHER HYPOTHESIS.

EXAMPLE - SPEECH RECOGNITION

CHOICES ARE 0, 1, 2, ..., 9

\Rightarrow MULTIPLE HYPOTHESIS TESTING
PROBLEM OR

PATTERN RECOGNITION OR
CLASSIFICATION

TO SOLVE ALL THESE PROBLEMS WE
STATISTICAL HYPOTHESIS TESTING
THEORY

MATHEMATICAL DETECTION PROBLEM

GIVEN DATA SET $\{x[0], x[1], \dots, x[N-1]\}$
 FIND $T(x[0], x[1], \dots, x[N-1])$ AND
 BASED ON T , MAKE A DECISION

EXAMPLE : DE LEVEL IN WHITE GAUSSIAN
 NOISE (WGN), $N=1$

OBSERVE EITHER $x[0] = w[0]$ (NOISE)
 OR $x[0] = A + w[0]$
 $= 1 + w[0]$
 (SIGNAL + NOISE)

HOW TO CHOOSE BASED ON VALUE OF
 $x[0]$ OBSERVED?

NOTE : $E(x[0]) = 0$ NOISE
 1 SIGNAL + NOISE

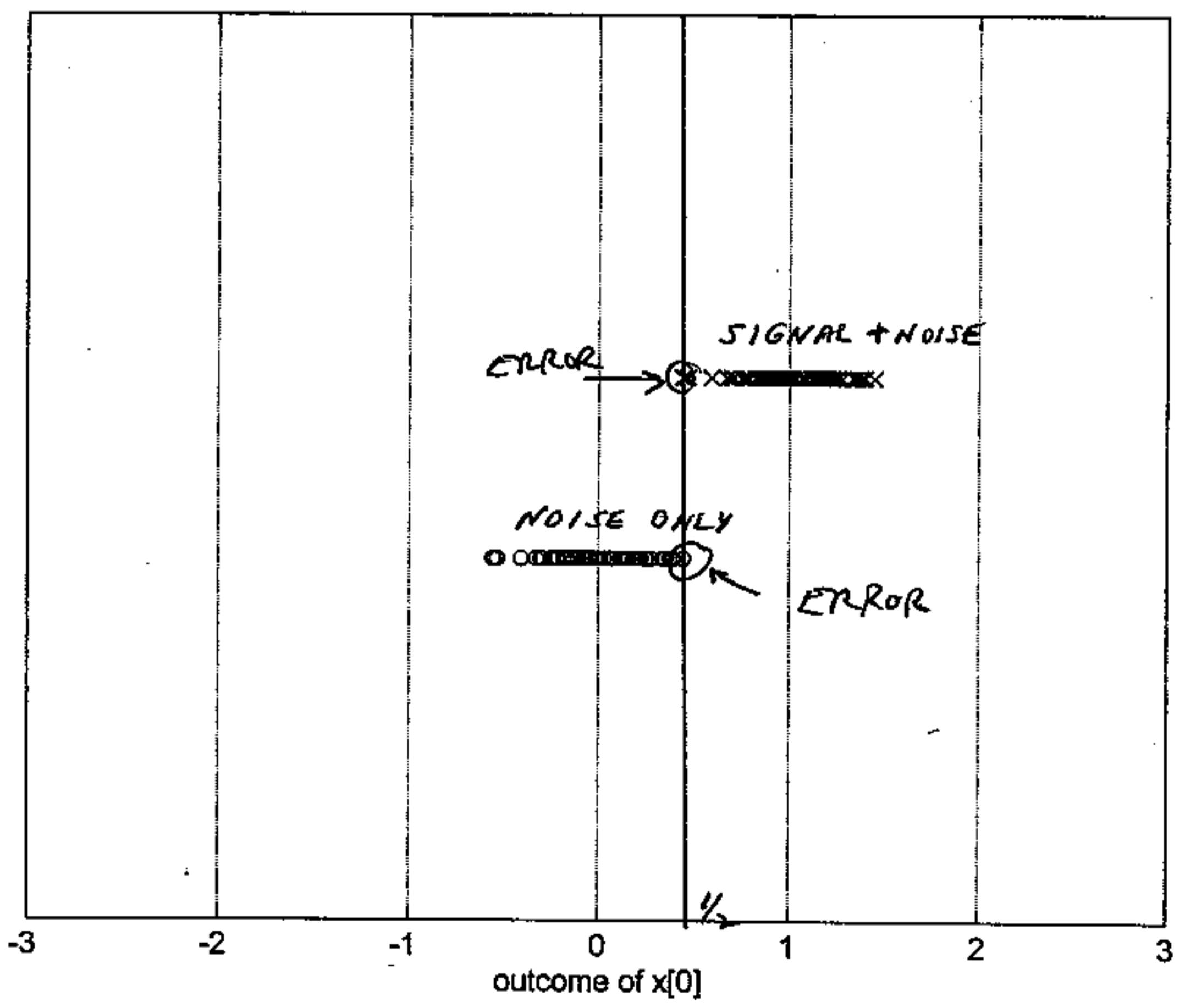
DETECTOR CHOOSES SIGNAL PRESENT IF

$$x[0] > \frac{1}{2}$$

GOOD DETECTOR? WHEN WILL WE BE
 WRONG?

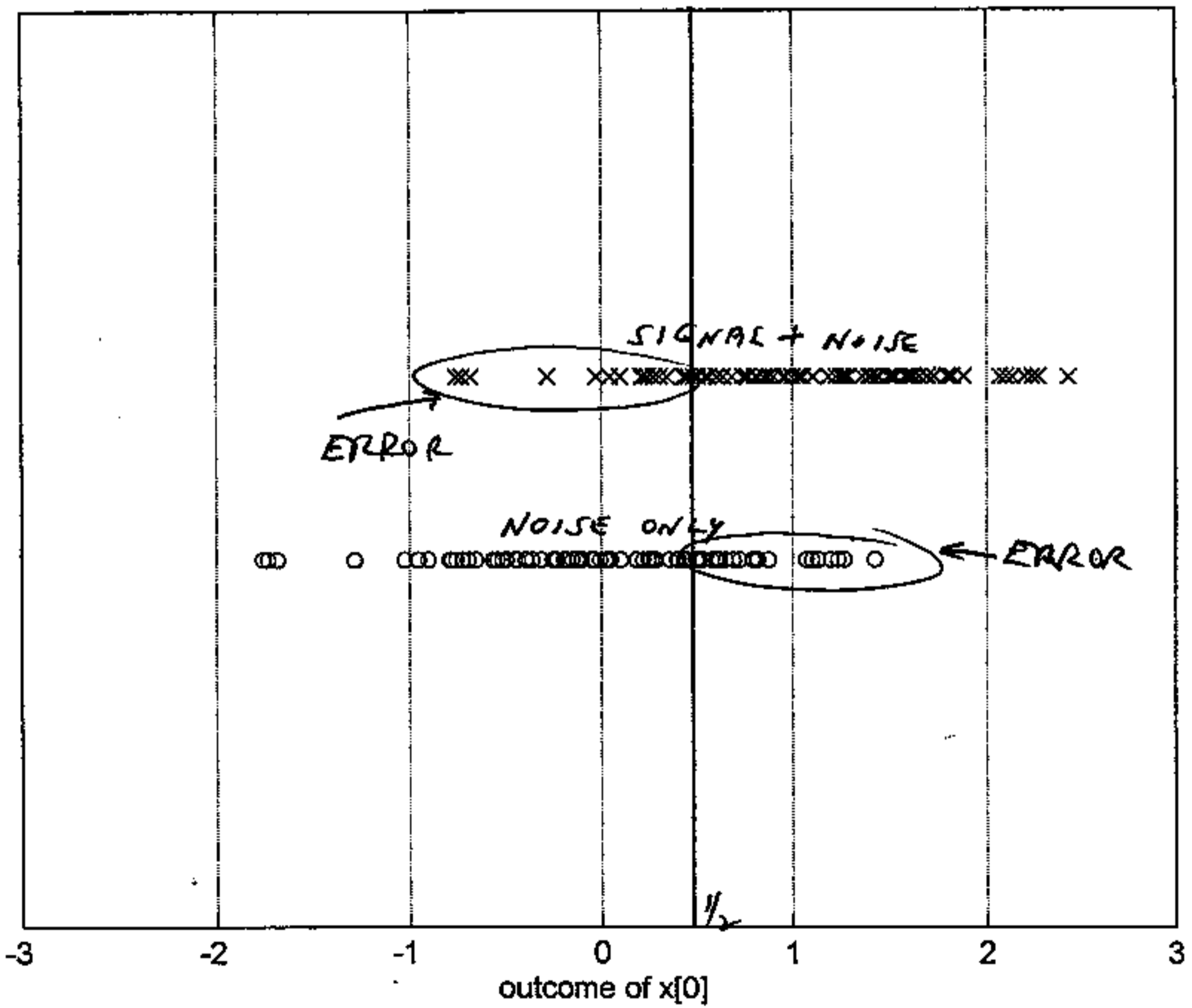
100 REALIZATIONS OF $x[0]$

$$\sigma^2 = 0.05$$



100 REALIZATIONS OF $x[0]$

$$\sigma^2 = 0.5$$



SEE FIG. 1.5 FOR HISTOGRAMS

PERFORMANCE DEPENDS ON SEPARATION
BETWEEN PDFS.

↑ DISCRIMINATION

LET $H_0 = \text{NOISE ONLY}$
 $H_1 = \text{SIGNAL + NOISE}$ } HYPOTHESES

$$H_0 : x(t) = w(t)$$

$$H_1 : x(t) = 1 + w(t)$$

PDFS ARE

$$p(x(t); H_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2(t)}$$

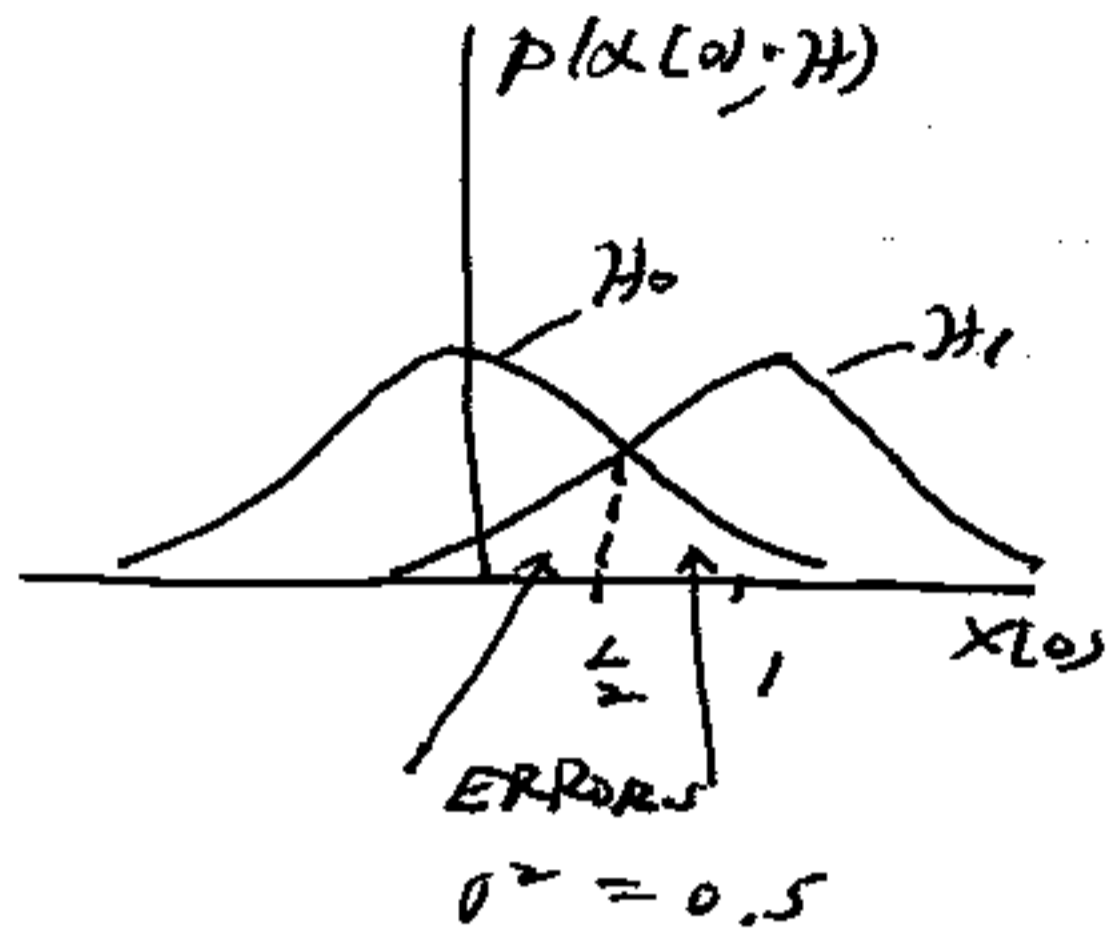
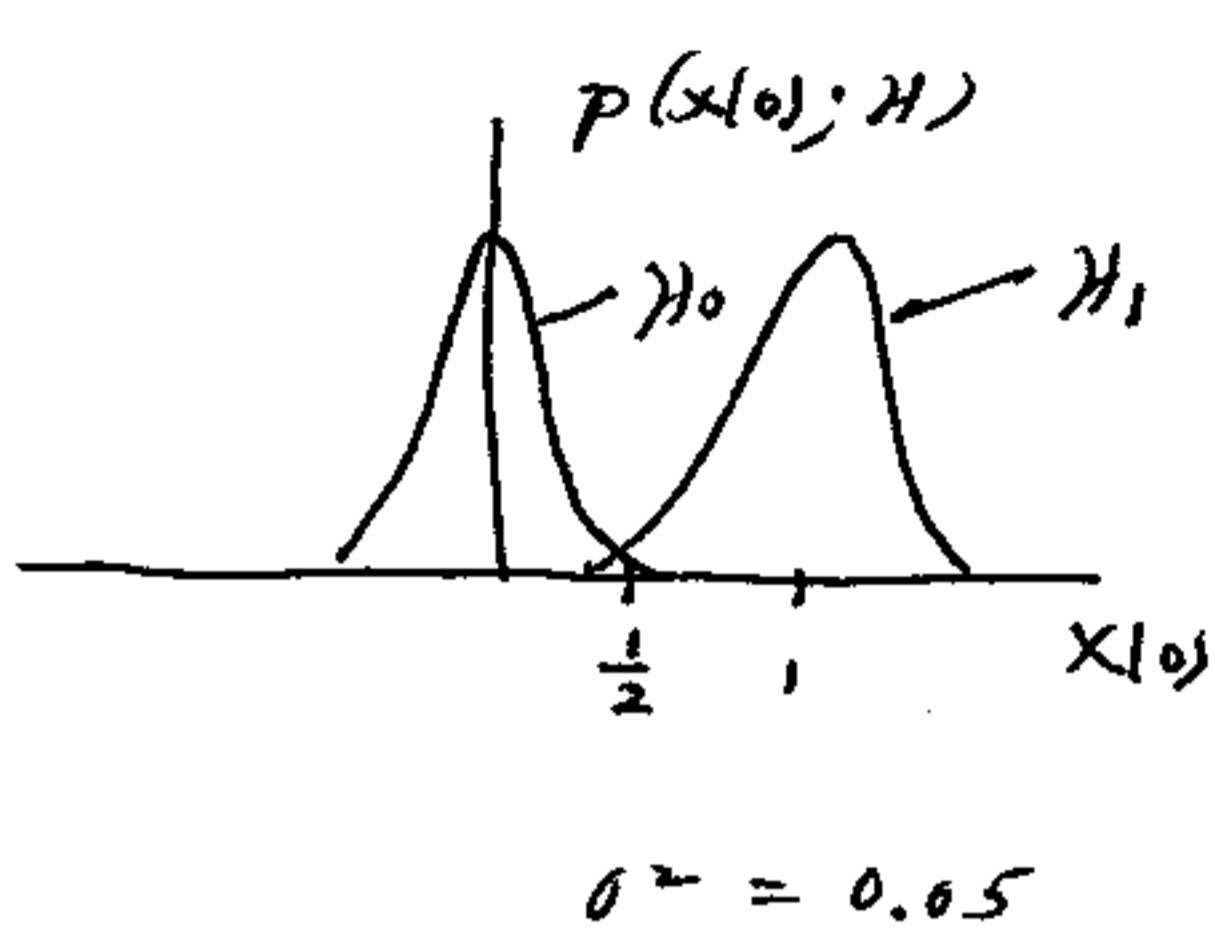
$$p(x(t); H_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} (x(t)-1)^2}$$

OR

$$x(t) \sim N(0, \sigma^2) \quad \text{UNDER } H_0$$

$$N(1, \sigma^2) \quad \text{" } H_1$$

SEE FIG. 1.6



ALTERNATIVE VIEWPOINT: FAMILY OF PDFS

$$p(x|0; A) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x|0-A)^2}$$

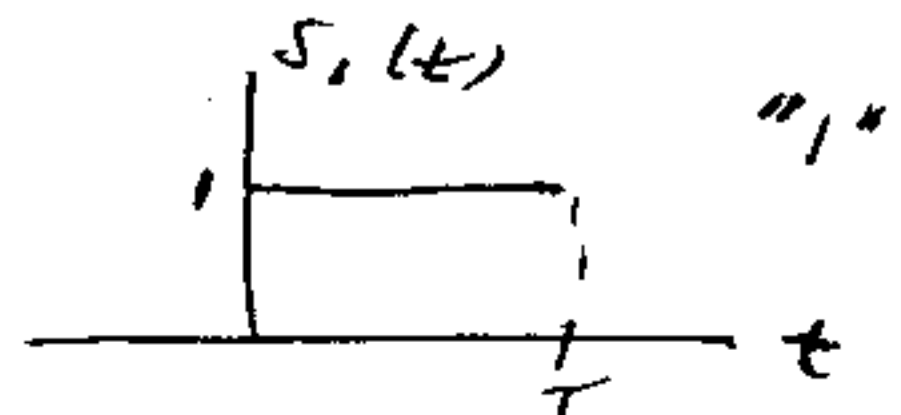
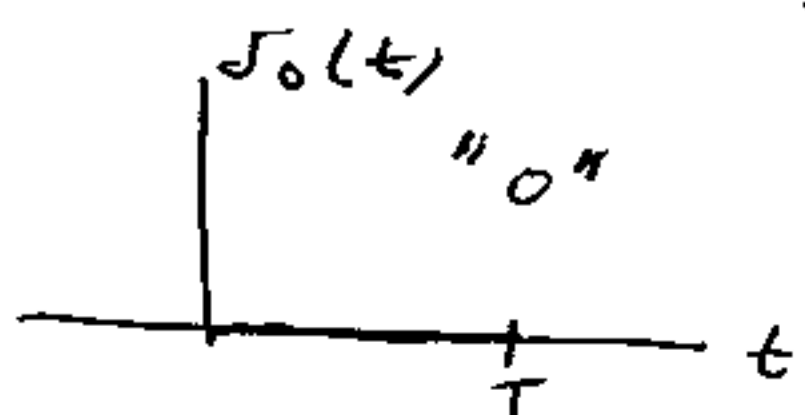
$-\infty < A < \infty$

$H_0 : A = 0$
 $H_1 : A = 1$

} TERMED A PARAMETER TEST

SOMETIMES WE THINK OF H_0, H_1 AS RANDOM EVENTS.

EXAMPLE : ON-OFF KEYED COMMUNICATION SYSTEM



FOR A BIT STREAM ... 0100110101100...
USUALLY HAVE HALF "0"'S AND HALF
"1"'S \Rightarrow

H_0 : "0" WITH PROB. = $\frac{1}{2}$
 H_1 : "1" " " " "

OR $P_r \{H_0\} = \frac{1}{2}$
 $P_r \{H_1\} = \frac{1}{2}$

IN THIS CASE PDF DENOTED AS

$p(x|0) | H_0$, $p(x|1) | H_1$
 \uparrow \uparrow

CONDITIONAL PDFS

ANALOGOUS TO CLASSICAL VS.
BAYESIAN ESTIMATION

DETECTION HIERARCHY

SEE TABLE 1.1.

NOISE

EASY-GOOD PERFORMANCE	GAUSSIAN	GAUSSIAN	NON GAUSSIAN	NON GAUSSIAN
	KNOWN PDF	UNKNOWN PDF	KNOWN PDF	UNKNOWN PDF
5 D E T E R M I N I S T I C K N O W N	4	9	10	*
1 D E T E R M I N I S T I C K U N K N O W N	7	9	10	*
N A L R A N D O M K N O W N P D F	5	9	*	*
R A N D O M U N K N O W N P D F	8	*	*	*

HARD-POOR PERFORMANCE

* NOT DISCUSSED (BEYOND SCOPE OF TEXT)

TABLE 1.1 - HIERARCHY OF DETECTION PROBLEMS AND CHAPTERS WHERE DISCUSSED

READ CHAPTER 2 - WILL COVER
SOME PARTS LATER

CHAPTER 3 - DECISION THEORY

EXAMPLE : OBSERVE $X(0)$ FROM EITHER
 $N(0, 1)$ OR $N(1, 1)$
 ↑ ↑
 MEAN VARIANCE
NORMAL OR
GAUSSIAN

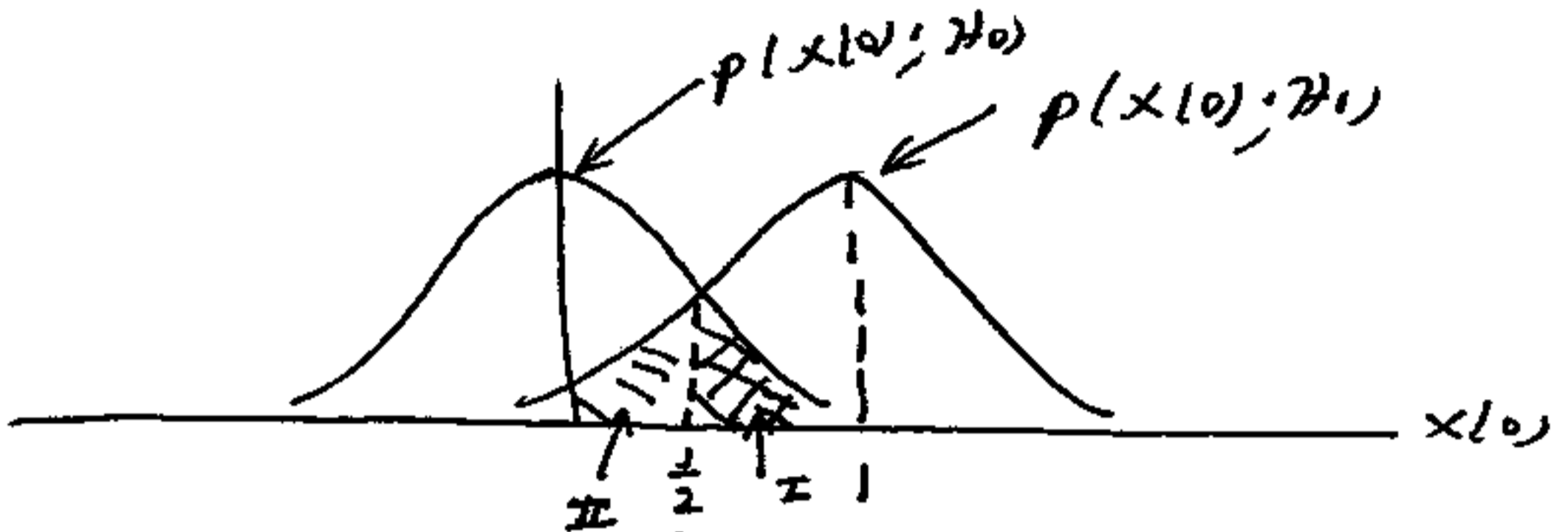
EQUIVALENT
→ $H_0 : \mu = 0$ PARAMETER TEST
 $H_1 : \mu = 1$
OR
→ $H_0 : X(0) = W(0)$
 $H_1 : X(0) = 1 + W(0)$

WHERE $W(0) \sim N(0, 1)$
 ↑
 DISTRIBUTED ACCORDING TO

IN STATISTICS $H_0 =$ NULL HYPOTHESIS
 $H_1 =$ ALTERNATIVE "

A REASONABLE APPROACH DECIDES H_1
IF $X(0) > 1/2$

$\frac{1}{2} = \underline{\text{THRESHOLD}}$



← TYPE II ERROR → | ← TYPE I ERROR →

ERRORS - DECIDE H_1 BUT H_0 IS TRUE - I
 " H_0 BUT H_1 " " - II

TYPE I ERROR HAS PROBABILITY

$$P(H_0; H_0) = P_r \left\{ x|0 > \frac{1}{2}; H_0 \right\}$$

\uparrow \uparrow
 DECIDED TRUE

TYPE II ERROR HAS PROBABILITY

$$P(H_0; H_1) = P_r \left\{ x|0 < \frac{1}{2}; H_1 \right\}$$

CAN WE REDUCE ERRORS ?

TO REDUCE TYPE I NOTE THAT WE DECIDE H_1 IF $x(0) > 1/2$. HOW ABOUT IF WE DECIDE H_1 IF $x(0) > 1$?

AS THRESHOLD CHANGES ONE ERROR INCREASES WHILE OTHER DECREASES.

IN RADAR PROBLEM

H_0 : $x(0) = w(0)$ NO AIRCRAFT
 H_1 : $x(0) = 1 + w(0)$ AIRCRAFT PRESENT

TYPE I ERROR IS TO DECIDE H_1 WHEN H_0 TRUE OR A FALSE ALARM.

$$\Rightarrow P_{FA} = P(H_1; H_0)$$

WANT THIS SMALL, SAY $\alpha = 10^{-8}$. WHY?

CONSTRAIN $P_{FA} = \alpha$. GIVEN THIS ERROR PROBABILITY, WISH TO MINIMIZE TYPE II OR MAXIMIZE

$$1 - P(H_0; H_1)$$

↑ TYPE II

TYPE II ERROR IS TO DECIDE H_0
 (NO AIRCRAFT) WHEN H_1 IS TRUE
 (AIRCRAFT PRESENT) \Rightarrow THIS IS
 A MISS

$$P_M = P(H_0; H_1)$$

$$\begin{aligned} 1 - P_M &= 1 - P(H_0; H_1) \\ &= \text{PROBABILITY OF } \underline{\text{DETECTION}} \\ &= P_D \end{aligned}$$

SUMMARY : MAKES SENSE TO
MAXIMIZE P_D SUBJECT TO
 CONSTRAINT $P_{FA} = \alpha$.
 (NEYMAN - PEARSON APPROACH)

FOR PREVIOUS PROBLEM WE CAN ADJUST
 THRESHOLD OR DECIDE H_1 IF
 $x(o) > \gamma$
 TO CONSTRAIN P_{FA} .

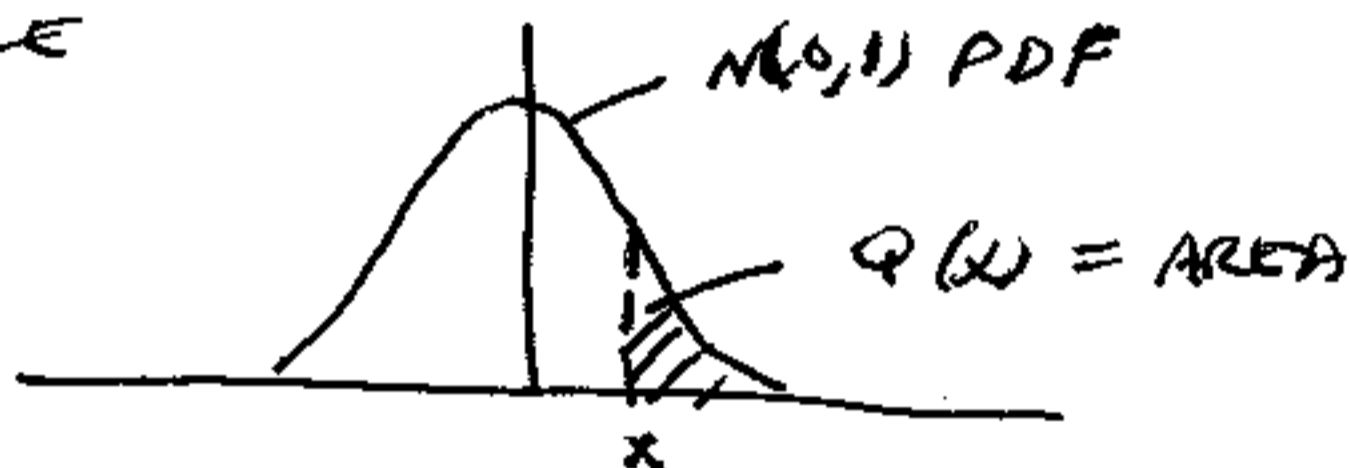
$$\begin{aligned} P_{FA} &= P(H_1; H_0) \\ &= P_r \{x(o) > \gamma; H_0\} \end{aligned}$$

$$= \int_{\gamma}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}}_{\text{UNDER } H_0 \text{ OR } X(0) \sim N(0,1)} dt$$

UNDER H_0 OR $X(0) \sim N(0,1)$

DEFINE $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$

RIGHT-TAIL PROBABILITY FOR $N(0,1)$ RANDOM VARIABLE



$$\Rightarrow P_{FA} = Q(\gamma)$$

FOR $P_{FA} = 10^{-3} \Rightarrow \gamma = 3$ (FROM TABLE)

DECIDE H_1 IF $X(0) > 3$. TO DETERMINE P_D

$$\begin{aligned} P_D &= P(H_1; H_1) \\ &= P\{X(0) > 3; H_1\} \end{aligned}$$

$$= \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-1)^2} dt$$

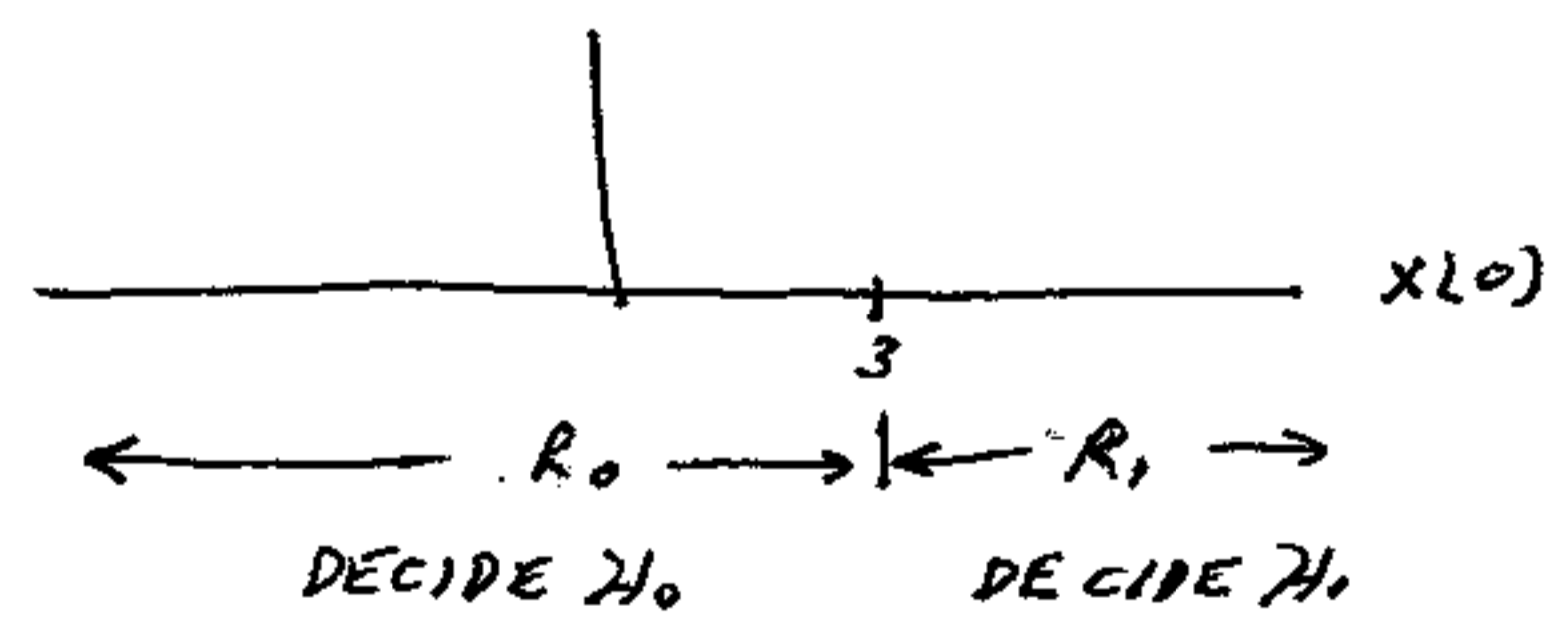
$$= Q(\gamma-1) = Q(2) = 0.023$$

NOT TOO GOOD - USUALLY WANT $P_D \geq 0.9$. BETTER DETECTOR?

NEYMAN-PEARSON LEMMA

GENERAL OPERATION OF DETECTOR -
GIVEN $\{x(0), x(1), \dots, x(N-1)\}$ MAP
INTO A DECISION

PREVIOUS EXAMPLE



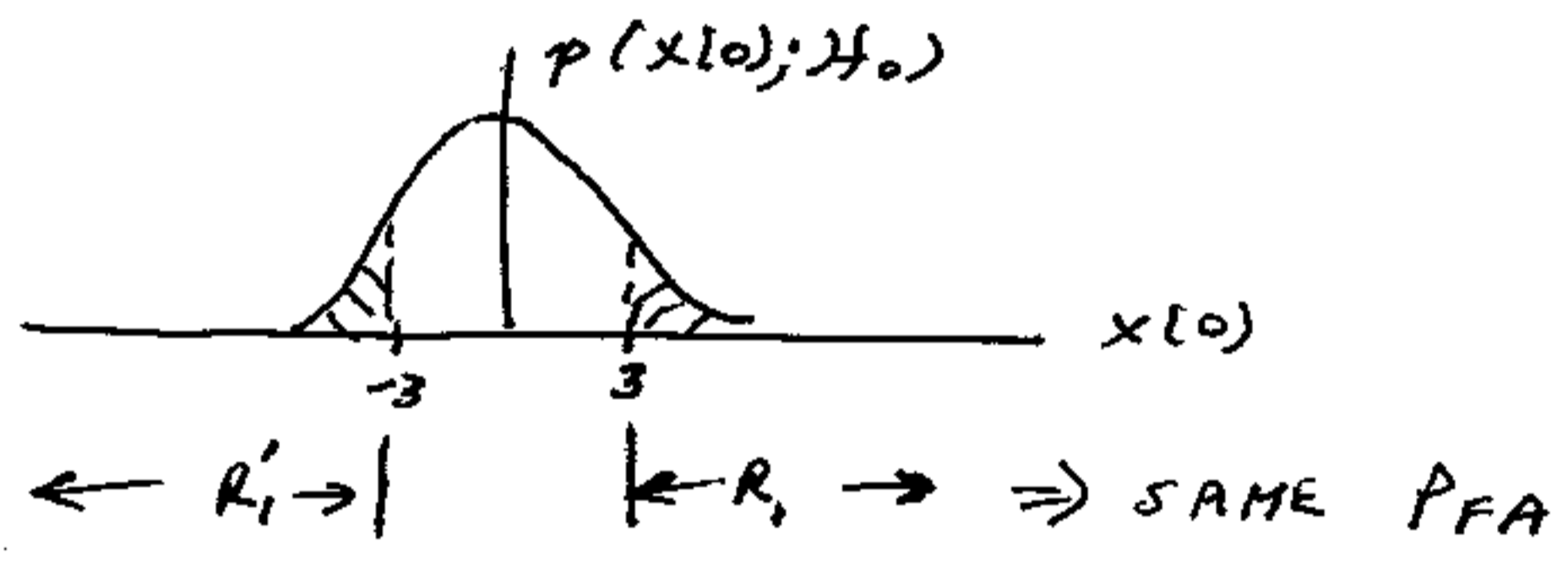
IN GENERAL

$$R_1 = \{x : \text{DECIDE } H_1\}$$

CALLED CRITICAL REGION

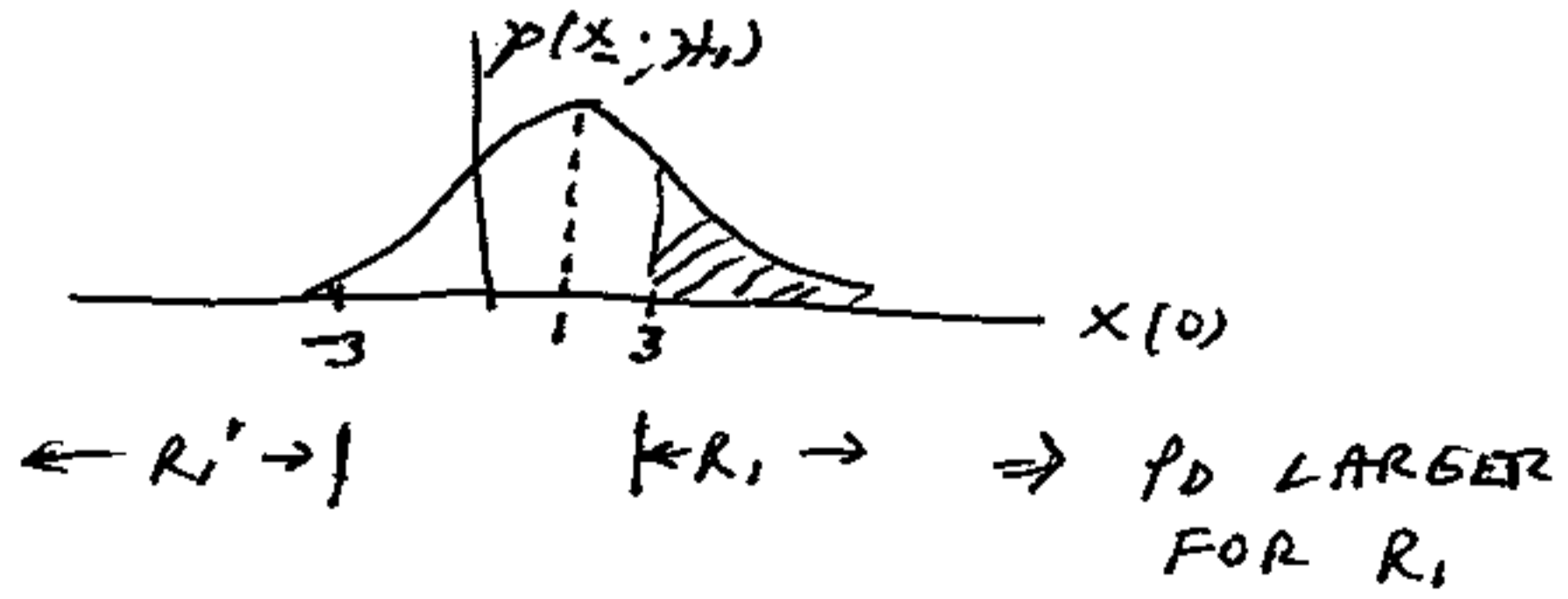
HENCE $P_{FA} = P\{\text{DECIDE } H_1; H_0\}$
 $= \int_{R_1} p(x; H_0) dx$

ONCE R_1 IS CHOSEN P_{FA} IS FIXED.
 MANY CRITICAL REGIONS EXIST
 WITH SAME P_{FA} . CHOOSE BEST ONE.



CLEARLY R_1 IS BETTER \Rightarrow LARGER P_D

$$P_D = \int_{R_1} p(x; H_1) dx$$



NEYMAN-PEARSON TELLS US HOW
 TO CHOOSE BEST R_1 .